Central Extension of Current Algebras on S^3, and the Central Extension of the Lie Algebra of Polynomial Type Infinitesimal Automorphisms on S^3

Tosiaki Kori

An affine Lie algebra is a 1-dimensional central extension of a simple Lie algebra with the *Laurent polynomial coefficients*. We develop an analogy for the current algebra on S^3. We introduce the algebra of *Laurent polynomial type harmonic spinors* $\mathcal{L}$ on S^3. Then we introduce a triple of 2-cocycles on $\mathcal{L}$. A central extension by $\mathbf{C}^3$ of the simple Lie algebra with the coefficients $\mathcal{L}$ is obtained.

Virasoro algebra is a 1-dimensional central extension of the Lie algebra of (complexified) vector fields over S^1. Khesin-Kravchenko and Radul introduced a 2-cocycle on the algebra ψDO of Pseudo-differential operators over a manifold. We apply their method to the Lie algebra of polynomial coefficient vector fields over S^3. The latter is a subalgebra of ψDO. We obtain an analogy of Virasoro algebra on S^3. We shall show the table of 2-cocycles of the basic vector fields on S^3.

This lecture consists as follows:

1. (After Kac's Bombay lecture note) Introduction to the central extension of the complex loop algebra on S^1 and the central extension of the Lie algebra of complex vector fields on S^1.

2. Classical analysis on Fourier analysisi on S^1, Cauchy operator and Laurent polynomials

3. A parallel analysis on S^3; Dirac operator on $\mathbf{R}^4$, the separation of variable method of the boundary Dirac operator, and the Laurent polynomial type spinors

4. Central extension of Current algebra on S^3; = central extension of the complex Lie algebra with Laurent polynomial type spinor coefficients by $\mathbf{C}^3$.

5. Algebra of polynomial coefficient vector fields on S^3: Witt algebra, and its central extension.

6. (i) The table of structure constants of the Witt algebra on S^3 and (ii) the table of the basic 2-cocycles (there are nine) on it.

As for the detailed explanation of (3) and (4) the readers can refer my previous papers (ask me (kori@waseda.jp)). As for the long calculations of the table of 2-cocycles the detailed calculations are available from me.

Preparatory

Lemma 0.1. *Let M be a Lie algebra over* $\mathbf{C}$ *and*

$$\tilde{M} = \{x + \lambda a; \quad x \in M,\, \lambda \in \mathbf{C}\}$$

with a an indefinite number.
Let $c : M \times M \longrightarrow \mathbf{C}$ *be a bilinear map satisfying*

$$c(x,y) = -c(y,x) \quad x, y \in M$$

$$c([x,y],\, z) + c([y,z],\, x) + c([z,x],\, y) = 0 \quad x, y, z \in M$$

(c is called 2-cocycle on M).
Then the Lie bracket

$$[x + \lambda a,\, y + \mu a] = [x,y] + \, c(x,y)a$$

makes $\tilde{M}$ into a Lie algebra.

$\tilde{M}$ is a central extension of M, i.e. there is a surjective homomorphism

$$\theta : \tilde{M} \ni x + \lambda a \longrightarrow x \in M$$

such that dim $ker\,\theta = 1$ and $ker\,\theta$ lies in the center of $\tilde{M}$; $[k, x + \lambda a] = 0$ for $\forall k \in ker\,\theta$ and $\forall x + \lambda a \in \tilde{M}$.

1 Loop algebras and central extensions

1.

$L = \mathbf{C}[t, t^{-1}]$: the Laurent polynomials in t:

$$\sum_{n \in \mathbf{Z}} a_n t^n, \quad t \in \mathbf{C}$$

finitely many $a_n \neq 0$.

The residue $Res : L \longrightarrow \mathbf{C}$ is defined by $Res\left(\sum a_n t^n\right) = a_{-1}$.

2.

$\mathfrak{g}, [\,,\,]$: a simple finite dimensional Lie algebra.

$L\mathfrak{g} = L \otimes_{\mathbf{C}} \mathfrak{g}$.

$L\mathfrak{g}$ may be made into a Lie algebra in a unique way

$$[\, f \otimes x,\, g \otimes y\,] = fg \otimes [x, y], \quad f, g \in L,\, x, y \in \mathfrak{g}.$$

Let $\langle\,,\,\rangle$ be the invariant bilinear form on $\mathfrak{g}$. Deine a bilinear form

$$\langle\,,\,\rangle_t \,:\, L\mathfrak{g} \times L\mathfrak{g} \longrightarrow L$$

by

$$\langle f \otimes x,\, g \otimes y \rangle_t = fg \otimes \langle x, y \rangle$$

Lemma 1.1. *The function* $c: L\mathfrak{g} \times L\mathfrak{g} \longrightarrow \mathbf{C}$ *defined by*

$$c(\xi,\, \eta) = \, Res\, \langle \frac{d\xi}{dt},\, \eta\rangle$$

is a 2-cocycle on $L\mathfrak{g}$.

Proof. To show that c is anticommutative it is sufficient to verify that

$$c(t^m \otimes x,\, t^n \otimes y) = -c(t^n \otimes y,\, t^m \otimes x).$$

Now

$$\begin{aligned} c(t^m \otimes x,\, t^n \otimes y) \;&=\; Res.\langle mt^{m-1} \otimes x, t^n \otimes y\rangle_t = Res.\,(mt^{n+n-1}\langle x, y\rangle) \\ &= \begin{cases} m\langle x, y\rangle & \text{if} m+n=0 \\ 0, & \text{if } m+n \neq 0 \end{cases} \end{aligned}$$

The anticommutativity follows. Jacobi identity is proved similarly. □

We may therefore construct the 1-dimensional central extension

$$\widehat{L\mathfrak{g}} \;=\; L\mathfrak{g} \oplus \mathbf{C}a$$

with Lie multiplication

$$[\xi + \lambda a\,,\, \eta + \mu a\,] = [\xi, \eta] + c(\xi, \eta)a\,, \quad \forall \xi, \eta \in L\mathfrak{g}\,,\; \lambda, \mu \in \mathbf{C}.$$

2 Loop algebras and central extensions

1.

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$$\langle\ ,\ \rangle_t\ :\ L\mathfrak{g} \times L\mathfrak{g} \longrightarrow\ L$$

by

$$\langle f \otimes x,\ g \otimes y \rangle_t\ = fg \otimes \langle x, y \rangle$$

Lemma 2.1. *The function* $c: L\mathfrak{g} \times L\mathfrak{g} \longrightarrow \mathbf{C}$ *defined by*

$$c(\xi, \eta) = Res \langle \frac{d\xi}{dt}, \eta \rangle$$

is a 2-cocycle on $L\mathfrak{g}$.

Proof. To show that c is anticommutative it is sufficient to verify that

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We may therefore construct the 1-dimensional central extension

$$\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{C}a$$

with Lie multiplication

$$[\xi + \lambda a\,,\, \eta + \mu a\,] = [\xi, \eta] + c(\xi, \eta)a\,, \quad \forall \xi, \eta \in L\mathfrak{g}\,,\ \lambda, \mu \in \mathbf{C}.$$

The Euler derivation $t\frac{d}{dt}$ acts on $\widehat{L\mathfrak{g}}$ as an outer derivation and kills a:

$$t\frac{d}{dt}(\xi + \lambda a) = t\frac{d\xi}{dt}\,, \quad \xi \in L\mathfrak{g}$$

Then adjoining a derivation d to $\widehat{L\mathfrak{g}}$ we have the Lie algebra $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{C}a \oplus \mathbf{C}d,$$

with the bracket defined as follows:

$$\begin{aligned}&[t^k \otimes x \oplus \lambda c \oplus \nu d,\, t^l \otimes y \oplus \lambda_1 c \oplus \nu_1 d\,]\\ &= (t^{k+l} \otimes [x, y] + \nu l t^l \otimes y - \nu_1 k t^k \otimes x) \oplus k\delta_{k,-l}(x|y)c\,,\end{aligned}$$

for $(x, y \in \mathfrak{g},\ \lambda, \nu, \lambda_1, \nu_1 \in \mathbf{C})$.

In brief an affine Lie algebra is a central extension of a simple Lie algebra with the *Laurent polynomial coefficients.*

To develop an analogy for the current algebra on S^3 we introduce the algebra of *Laurent polynomial type harmonic spinors* on S^3. Then we introduce a triple of 2-cocycles on this algebra.

3 Lie algebra of complex vecor fields on S^1

1. $Vect(S^1)$: the Lie algebra of *real* vector fields on S^1:

$$Vect(S^1) \ni f(\theta)\frac{\partial}{\partial\theta}\,,\; f \in C^{\infty}(S^{)}$$

The LIe bracket of vector fields is:

$$[f(\theta)\frac{\partial}{\partial\theta}\,,\; g(\theta)\frac{\partial}{\partial\theta}\,] = (fg' - f'g)(\theta)\frac{\partial}{\partial\theta}$$

A real basis is provided by the vector fields

$$\frac{\partial}{\partial\theta},\; \cos(n\theta)\frac{\partial}{\partial\theta},\; \sin(n\theta)\frac{\partial}{\partial\theta}, \quad n = 1, 2, \cdots$$

Remark 3.1. (no convergence discussion, so $f(\theta), g(\theta)$ are any trigonometric polynomials.

2. The complexfication of $Vect(S^1)$:

$$\mathcal{D} = \left\{ d_n = ie^{in\theta}\frac{\partial}{\partial\theta} = -z^{n+1}\frac{d}{dz}\,; \quad n \in \mathbf{Z},\; z = e^{i\theta} \right\}$$

The LIe bracket of vector fields is:

$$[\, d_m\,,\, d_n\,] = (m-n)d_{m+n}\,, \quad m, n \in \mathbf{Z}$$

Remark 3.2. The Lie algebra $Vect(S^1)$ is the Lie algebra $Diff_o(S^1)$ of orientation preserving diffeomorphisms of S^1:

$\gamma \in Diff_o(S^1)$ acts on $C^\infty(S^1, \mathbf{C})$;

$$(\gamma \cdot f)(z) = f(\gamma^{-1} z).$$

For $\gamma \sim Id$, we have

$$\gamma(z) = z + z\epsilon(z) = z + \sum_{n=-\infty,\, n\neq 0}^{\infty} \epsilon_n z^{n+1},$$

$\epsilon(z) = z + \sum_{n=-\infty,\, n\neq 0}^{\infty} \epsilon_n z^{n+1}$ is the Laurent (Fourier) series.

Since

$$\gamma^{-1}(z) = z - \sum_{n=-\infty,\, n\neq 0}^{\infty} \epsilon_n z^{n+1},$$

we have

$$(\gamma \cdot f)(z) = f(z - \sum_{n=-\infty,\, n\neq 0}^{\infty} \epsilon_n z^{n+1}) = (1 + \sum_n \epsilon_n d_n) f(z)$$

with $d_n = -z^{n+1} \frac{d}{dz}$.

Hence the d_n's form a topological basis of the complexication of the Lie algebra $\mathcal{D}$.

Central extensions of $\mathcal{D}$: the Virasoro algebra

Suppose we have a 2-cocycle $c(m,n) \in \mathbf{C}$, $m,n \in \mathbf{Z}$.

Then the central extension $\tilde{\mathcal{D}}$ of $\mathcal{D}$ by a 1-dimensional center $\mathbf{C}a$ is obtained:

$$\tilde{\mathcal{D}} = \mathcal{D} \oplus \mathbf{C}a$$

$$[\, d_m,\, d_n\,] = (m-n)d_{m+n} \,+\, c(m,n)a\,, \quad [\, d_m,\, a] = 0\,. \tag{3.1}$$

- CALCULATIONS (Kac: Bombay Lecture or ...)

1. rescale:$d'_0 = d_0, \quad d'_n = d_n - \frac{c(0,n)}{n} a,\ n \neq 0$ to have

$$[d'_0, d'_n] = -nd'_n,\ n = 0, \pm 1, \cdots$$

   Suppose this relations

$$[d_0, d_n] = -nd_n \tag{3.2}$$

   from the first.

2. Jacobi identity for $d_0, d_m, d_n \Longrightarrow$

$$[d_0, [d_m, d_n]] = -(m+n)[d_m, d_n] \tag{3.3}$$

   (3.1)(3.2)(3.3) $\Longrightarrow$ $(m+n)c(m,n)a = 0 \Longrightarrow c(m,n) = \delta_{m,-n}c(m)$, so that

$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}c(m)a, \tag{3.4}$$

3. From the anticommutativity of Lie bracket it holds $c(m) = -c(-m)$.

4. Jacobi identity for d_l, d_m, d_n with $l + m + n = 0$ and (3.4) $\Longrightarrow$

$$(m-n)c(m+n) - (2n+m)c(m) + (n+2m)c(n) = 0 \tag{3.5}$$

   and

$$(m-1)c(m+1) = (2+m)c(m) - (1+2m)c(1) \tag{3.6}$$

5. Since $c(-m) = -c(m)$, $c(0) = 0$, eough to solve it for $m > 0$ only. (3.6) being linear recursion relation, the dimention of the solution space is at most 2.

   $c(m) = \alpha m + \beta m^3$.

6. For $\beta = 0$ the extension becomes the trivial one

7. For $\beta \neq 0$ we may take

$$c(m) = \beta(m^3 - m)$$

We put $\beta = \frac{1}{12}$

Definition 3.3. *Virasoro algebra* is the Lie algebra Vir with basis

$$\{\, d_m,\ m \in \mathbf{Z};\ a\,\}$$

and the commutationrelations

$$[d_m\,,\,a\,] = 0 \tag{3.7}$$

$$[\,d_m\,,\,d_n\,] = (m-n)d_{m+n} + \delta_{m,-n}\frac{(m^3-m)}{12}a \tag{3.8}$$

4 Classical Analysis on

$$\mathbf{C} \quad \simeq \mathbf{R}^2 \quad \simeq [0.\infty) \times S^1$$

$$z = x + iy \ \sim \begin{pmatrix} x \\ y \end{pmatrix} \ \sim re^{i\phi}\,.$$

- $U(1) \ni e^{i\theta} \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$ acts on $\mathbf{R}^2$:

$$z = x + iy = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow g \cdot z = \begin{pmatrix} x\,\cos\theta - y\,\sin\theta \\ x\,\sin\theta + y\,\cos\theta \end{pmatrix}$$

- $SO(2)$ acts on $C^\infty(\mathbf{R}^2)$:

$$(R_g F)(z) = F(g^{-1}z) = F(x\,\cos\theta + y\,\sin\theta\,, -x\,\sin\theta + y\,\cos\theta)$$

- $$so(2) = \{X;\ 2 \times 2 \text{ matrix}/\mathbf{R}\,,\ X + {}^tX = 0\,,\ tr.X = 0\}$$

  Infinitesimal action of $so(2)$ on $\mathbf{R}^2$:

$$(dR)_e :\ u(1) \sim\ so(2)\ \longrightarrow\ Vect(\mathbf{R}^2)$$

$$so(2) \ni \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \longrightarrow (dR)_e(\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}) \ = \ -\frac{\partial}{\partial\theta}$$

In fact.

$$dR(\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix})F(x,y) \;=\; \frac{d}{dt}|_{t=0}\, R_{e^{it}}F(x,y)$$

$$= (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})F \;=\; -\frac{\partial}{\partial \theta}F\,.$$

$$\tfrac{d}{dt}\, R_{e^{it}}F(x,y) = \tfrac{d}{dt}\, F(x\cos\, t + y\sin\, t\,,\, -x\sin t + y\cos t)$$

$$= \tfrac{\partial F}{\partial x}\,\tfrac{d}{dt}(x\,\cos\, t + y\,\sin\, t) + \tfrac{\partial F}{\partial y}\,\tfrac{d}{dt}(-x\sin t + y\cos t),$$

$$\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation vector field around $0 \in \mathbf{R}^2$ and
is **a basis of** $Vect(S^1)$.

4.1 Cauchy operator $\overline{\partial}$

- $$\mathbf{C} \ni z = x + iy = r\,e^{i\theta} = \cos\theta + i\sin\theta$$
  $$x = \frac{1}{2}(z + \overline{z}), \qquad y = \frac{1}{2i}(z - \overline{z})$$

$$\begin{aligned} \overline{\partial} : \;&=\; \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ &=\; \frac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + i\,\frac{1}{r}\,\frac{\partial}{\partial \theta}\right). \end{aligned}$$

- **Eigenvalue problem on $S^1 = \{z = e^{i\theta}\}$ of the boundary Cauchy operator $\frac{\partial}{\partial\theta}$.**

$$-i\,\frac{\partial}{\partial\theta}\,\phi(\theta) \;=\; \lambda\phi(\theta)$$

is solved by the eigenvalues:

$$\lambda = 0,\ \pm 1,\ \pm 2,\ \cdots,\pm n,\ \cdots$$

and eigen functions:

$$\phi_k(\theta) \;=\; \frac{1}{\sqrt{2}}\,e^{ik\theta}\,, \quad k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

with the normalization condition:

$$\int_{-\pi}^{\pi} \phi_j(\theta)\,\overline{\phi_k(\theta)}\,d\theta \;=\; \delta_{jk}$$

- **Fourier expansion**

  $\forall f \in C^(S^1)$

$$f(\theta) = \sum_{k=-\infty}^{\infty} c_k \, \phi_k(\theta)$$

$$c_k = \int_{-\pi}^{\pi} f(\theta) \, \overline{\phi_k(\theta)} \, d\theta.$$

We shall show

- "Fourier expansion" $\Longleftrightarrow$ " Laurent expansion ".
- basic functions $e^{\pm ik\theta}$ $\Longleftrightarrow$ basic functions $z^{\pm n}$

Separation of variables & holomorphic extension

- Given $f(\theta) = \sum_{k=-\infty}^{\infty} c_k\, \phi_k(\theta)$ on S^1; $c_k = \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta$.

  Find F on $D^2 = \{|z| \le 1\}$ such that

$$\begin{cases} \overline{\partial} F &= 0, \quad |z| < 1 \\ F &= f\,, \quad |z| = 1, \end{cases} \qquad z = re^{\imath\theta},\ 0 \le r < 1,\ -\pi < \theta \le \pi$$

- *The solution obtained as follows.*

  Let the extension have the formula

$$F(re^{i\theta}) \;=\; \sum_{k=-\infty}^{\infty} a_k\, R_k(r)\phi_k(\theta).$$

  Since $\quad \overline{\partial} F \;= \frac{1}{2} e^{i\theta}\,\left(\frac{\partial}{\partial r} + i\,\frac{1}{r}\,\frac{\partial}{\partial\theta}\right) F = \;0\;$,
  it holds that

$$\begin{aligned} &\textstyle\sum_k\, a_k\,(\, \frac{dR_k}{dr}\, \phi_k(\theta) \;+\; \frac{i}{r}\, R_k\, \frac{d\phi_k}{d\theta}\,) \\ =& \textstyle\sum_k\, a_k\,(\, \frac{dR_k}{dr} \;-\; \frac{k}{r}\, R_k\,)\phi_k \;= 0. \end{aligned}$$

- From the conditions: F: regular at $z = 0$, ϕ_k: linearly independent, and $F = f$ on $|z| = 1$, we have

$$\begin{cases} a_k = 0, \quad \forall k \le -1, \\ a_k R_k(r) = c_k\, r^k \end{cases}$$

$a_k = 0, \qquad \forall k \leq -1$ and $a_k R_k(r) = c_k \, r^k.$

$$F(z) \;= F(re^{i\theta}) = \; \textstyle\sum_{k=0}^{\infty} \, c_k \, r^k \phi_k(\theta) \; = \textstyle\sum_{k=0}^{\infty} \, c_k \, z^k$$

$$c_k = \textstyle\int_{-\pi}^{\pi} \, f(\theta) \, e^{-ik\theta} \, d\theta$$

$$\begin{aligned} F(z) \;&=\; \sum_{k=0}^{\infty} c_k z^k \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sum_{k=0}^{\infty} e^{-ik\theta} z^k d\theta \\ &=\; \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta) \left(\sum_{k=0}^{\infty} \overline{\zeta}^k z^k \right) \overline{\zeta} d\zeta \;=\; \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)\overline{\zeta}}{1 - \overline{\zeta}\, z} d\zeta \\ &=\; \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta \end{aligned}$$

Cauchy integral formula on $|z| \leq 1$!

- **exterior Dirichelet problem**

  Given $f(\theta) = \sum_{k=-\infty}^{\infty} c_k\, \phi_k(\theta)$ on S^1; $c_k = \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta$.
  Find F on $D^c = \{|z| \geq 1\}$ such that

$$
\begin{cases}
\overline{\partial} F &= 0, \quad |z| > 1 \\
F &= f\,, \quad |z| = 1, \qquad\qquad z = re^{\imath\theta},\ 0 \leq r < 1,\ -\pi < \theta \leq \pi \\
F &\overset{|z|\longrightarrow\infty}{\longrightarrow} O(1/|z|).
\end{cases}
$$

  F is regular at $z = \infty$, ϕ_k is linearly independent, and $F = f$ on $|z| = 1$,
  it follows that

$$
\begin{cases}
a_k = 0, \qquad k \geq 1, \\
a_k R_k(r) = c_k\, r^k
\end{cases}
$$

$$F(z) = F(re^{i\theta}) = \sum_{-\infty}^{k=1} c_k\, r^k \phi_k(\theta) = \sum_{k=1}^{\infty} c_{-k}\, z^{-k}$$

$$
\begin{aligned}
F(z) &= \sum_{k=1}^{\infty} c_{-k} z^{-k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sum_{k=1}^{\infty} e^{ik\theta} z^{-k} d\theta \\
&= -\frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta) \left(\sum_{k=1}^{\infty} \zeta^{k-1} z^{-k} \right) d\zeta \\
&= -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta
\end{aligned}
$$

  exterior Cauchy integral formula on $|z| > 1$.

- **Laurent expansion** F holomorphic on $0 < |z| < \infty$ has the expansion

$$F(z) = \frac{1}{2\pi i}\int_{|\zeta|=r_2} \frac{f(\zeta)}{\zeta - z} d\zeta \; - \; \frac{1}{2\pi i}\int_{|\zeta|=r_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $r_1 < |z| < r_2$, $0 < \forall r_1 < \forall r_2$.

- $z^{\pm n}$: **a basis of holomorphic functions on** $0 < |z| < \infty$:

$$F(z) = \sum_{n=0} c_n z^n + \sum_{n=1} c_{-n} z^{-n}$$

- **Residue**

The coefficient c_{-1} of z^{-1} is called the *residue* of F at $z = 0$:

$$Res.F(0) = \frac{1}{2\pi i}\int_{|\zeta|=\epsilon} F(\zeta) d\zeta, \quad \forall \epsilon > 0.$$

- **Ingredients** consist of

$$\cdots, z^{-2}, z^{-1}, 1, z, z^2, \cdots;$$

$$\frac{\partial}{\partial\theta} = z\frac{d}{dz}\,|S^1;$$

$$\overline{\partial} = \frac{\partial}{\partial \bar{z}}$$

If we have the corresponding ingredients on $S^3 \subset \mathbf{C}^2$ we shall obtain "classical analysis" on S^3 and its calculations.

- [Clifford-Hamilton, Dirac-Pauli]

  - "functions " $\Longrightarrow$ matrix and then "spinors"

  - $$U(1) \sim SO(2) \Longrightarrow SU(2)/\pm I \sim SO(3)$$

5 Contemporary Classical Analysis on

$$\mathbf{C}^2 \simeq \mathbf{R}^4 \simeq [0.\infty) \times S^3$$

- (i) Basic polynomials on $S^3 \subset \mathbf{C}^2$
  (ii) Basic vector fields on $S^3 \subset \mathbf{C}^2$
  that comes from the representation of $SU(2) \simeq SO(3)$

- $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2) \sim SO(3)$ acts on $\mathbf{C}^2 \sim \mathbf{R}^4$:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow R_g z = \begin{pmatrix} az_1 - b\overline{z}_2 \\ az_2 + b\overline{z}_1 \end{pmatrix}$$

- $SU(2)$ acts on $C^\infty(\mathbf{C}^2)$:

$$(R_g F)(z) = F(z \cdot g) = F(az_1 - b\overline{z}_2\,, az_2 + b\overline{z}_1)$$

- **Basis of** $su(2) = \{X \in \mathfrak{gl}(2, \mathbf{C}) : {}^t\overline{X} + X = 0,\, tr.X = 0\}$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$ Pauli matrices.

- Infinitesimal action of $su(2)$: $(dR)_e \,:\, su(2) \longrightarrow Vect(S^3)$.

$$(dR_e)(X)F = \frac{d}{dt}|_{t=0} R_{\exp tX} F\,, \quad X \in su(2),$$

is given by

$$dR(\sigma_3) = -\sqrt{-1}\theta_0\,,\, dR(\sigma_2) = \sqrt{-1}\theta_1\,,\, dR(\sigma_1) = \sqrt{-1}\theta_2\,.$$

Where

$$\begin{aligned} \theta_1 &= e_+ + e_-\,, & \theta_2 &= \sqrt{-1}(e_+ - e_-)\,, \\ e_+ &= -z_2\tfrac{\partial}{\partial \bar{z}_1} + z_1\tfrac{\partial}{\partial \bar{z}_2}, & e_- &= -\bar{z}_2\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial z_2}\,. \\ \theta_0 &= \sqrt{-1}\theta = \sqrt{-1}(z_1\tfrac{\partial}{\partial z_1} + z_2\tfrac{\partial}{\partial z_2} & &-\bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}). \end{aligned}$$

We have the commutation relations;

$$[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta.$$

Lie algebras spanned by (e_+, e_-, θ) is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$.

- By the Euler angle coordinates (θ, ϕ, ψ) on S^3;

$$z_1 = \cos\frac{\theta}{2}\exp(\frac{\sqrt{-1}}{2}(\psi + \phi))\,,\; z_2 = \sqrt{-1}\sin\frac{\theta}{2}\exp(\frac{\sqrt{-1}}{2}(\psi - \phi))$$

we have the following expression:

$$\begin{aligned}
\theta_0 &= \frac{\partial}{\partial\psi}, \\
\theta_1 &= -\sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta\cos\psi\frac{\partial}{\partial\psi} \\
\theta_2 &= \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta\sin\psi\frac{\partial}{\partial\psi}
\end{aligned}$$

These are the **rotation vector fields** around $0 \in \mathbf{R}^4$,
and give **a basis of** $Vect(S^3)$.

6 Dirac operator

-

$$\mathbf{C}^2 \ni (z_1 = x_1 + ix_2, z_2 == x_3 + ix_4)$$

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4},$$

$$\sigma_k, \quad k = 1, 2, 3, 4, \quad \text{Pauli matrices}$$

The half Dirac operator has the expression:

$$\begin{aligned} D &= \begin{pmatrix} \frac{\partial}{\partial z_1} & -\frac{\partial}{\partial \bar{z}_2} \\ \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \bar{z}_1} \end{pmatrix} \\ &= \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\sigma_3 - i\frac{\partial}{\partial x_3}\sigma_2 - i\frac{\partial}{\partial x_4}\sigma_1 \end{aligned}$$

[Remember $\overline{\partial} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}$.]

- *Polar decomposition* of the Dirac operators D :

$$D = \gamma_+ \left(\frac{\partial}{\partial r} - \partial\!\!\!/ \right).$$

$\partial\!\!\!/$; the boundary Dirac operator , is given by

$$\partial\!\!\!/ = \frac{1}{r} \begin{pmatrix} -\frac{1}{2}\theta & e_+ \\ -e_- & \frac{1}{2}\theta \end{pmatrix}.$$

In the above

$$r = |z|.$$

$$\frac{\partial}{\partial r} = \frac{1}{2|z|}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \overline{z}_1\frac{\partial}{\partial \overline{z}_1} + \overline{z}_2\frac{\partial}{\partial \overline{z}_2}\right).$$

γ denote the Clifford multiplication of the radial vector $\frac{\partial}{\partial r}$:

$$\gamma = \gamma_+ \oplus \gamma_- : S^+ \oplus S^- \longrightarrow S^- \oplus S^+ ,$$

$$\gamma^2 = 1$$

[Remember; $\overline{\partial} = \frac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + i\,\frac{1}{r}\,\frac{\partial}{\partial \theta}\right)$.]

- **Eigenvalue problem on S^3 of the boundary Dirac operator $\partial\!\!\!/$.**

  We define the monomials:

$$v^k_{(l,m-l)} = (e_-)^k z_1^l z_2^{m-l}\,; \quad m = 0,1,2,\cdots, \quad l,k = 0,1,\cdots,m$$

  They play the same role on $S^3 \subset \mathbf{C}^2$ as the monomials $z^{\pm n}$ do on $S^1 \subset \mathbf{C}$:

  1. $v^k_{(l,m-l)}$ is a harmonic polynomial on $\mathbf{C}^2$; $\Delta v^k_{(l,m-l)} = 0$.
  2. $\left\{ \frac{1}{\sqrt{2}\pi} v^k_{(l,m-l)}\,;\, m = 0,1,\cdots, 0 \le k,l \le m \right\}$ forms a complete orthonormal system of $L^2(S^3)$.
  3. For each pair (m,l), $0 \le l \le m$, $H_{(m,l)} = \{v^k_{(l,m-l)}\,; 0 \le k \le m+1\}$ gives a $(m+1)$-dimensional *representation of $su(2)$ with the highest weight* $\frac{m}{2}$.

$$\begin{aligned} e_+ v^k_{(l,m-l)} &= -k(m-k+1) v^{k-1}_{(l,m-l)}, \\ e_- v^k_{(l,m-l)} &= v^{k+1}_{(l,m-l)}, \\ \theta v^k_{(l,m-l)} &= (m-2k) v^k_{(l,m-l)}\,. \end{aligned}$$

  4. Therefore the space of harmonic polynomials on $\mathbf{C}^2$ is decomposed by the right action of $su(2)$ into $\sum_m \sum_{l=0}^m H_{m,l}$.

-

  The boundary Dirac operator on the sphere $S^3 = \{r = 1\}$;

$$\partial\!\!\!/ | S^3 : C^\infty(S^3, S^+) \longrightarrow C^\infty(S^3, S^+)$$

  is a *self adjoint elliptic differential operator.*

- A basis of the space of even harmonic spinors;

$$\begin{aligned}
\phi^{+(m,l,k)}(z) &= \sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}} \begin{pmatrix} k v^{k-1}_{(l,m-l)} \\ \\ -v^k_{(l,m-l)} \end{pmatrix}, \\
\phi^{-(m,l,k)}(z) &= \sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}} \left(\frac{1}{|z|^2}\right)^{m+2} \begin{pmatrix} w^k_{(m+1-l,l)} \\ \\ w^k_{(m-l,l+1)} \end{pmatrix}., \\
m &= 0, 1, 2, \cdots; l = 0, 1, \cdots, m, \; k = 0, 1, \cdots, m+1,
\end{aligned}$$

  where

$$w^k_{(l,m-l)} = (-1)^k \frac{l!}{(m-k)!} v^{m-l}_{(k,m-k)} \, .$$

Proposition 6.1. *1.* $\phi^{+(m,l,k)}$ *is a harmonic spinor on* $\mathbf{C}^2$ *and* $\phi^{-(m,l,k)}$ *is a harmonic spinor on* $\mathbf{C}^2 \backslash \{0\}$ *that is regular at infinity.*

2. *On* $S^3 = \{|z| = 1\}$ *we have:*

$$\partial\!\!\!/ \phi^{+(m,l,k)} = \frac{m}{2} \phi^{+(m,l,k)}, \qquad \partial\!\!\!/ \phi^{-(m,l,k)} = -\frac{m+3}{2} \phi^{-(m,l,k)}.$$

3. *The eigenvalues of* $\partial\!\!\!/$ *are*

$$\frac{m}{2}, \quad -\frac{m+3}{2}; \quad m = 0, 1, \cdots,$$

and the multiplicity of each eigenvalue is equal to $(m+1)(m+2)$.

4. *The set of eigenspinors*

$$\left\{ \frac{1}{\sqrt{2}\pi} \phi^{+(m,l,k)}, \quad \frac{1}{\sqrt{2}\pi} \phi^{-(m,l,k)}; \quad m = 0, 1, \cdots, 0 \le l \le m, \, 0 \le k \le m+1 \right\}$$

forms a complete orthonormal system of $L^2(S^3, S^+)$.

- The restriction of $\mathbf{C}[\phi^{\pm}]$ to S^3 is an associative algebra generated by the spinors:

$$\phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ \phi^{+(1,0,1)} = \begin{pmatrix} z_2 \\ -\overline{z}_1 \end{pmatrix},$$

$$\phi^{-(0,0,0)} = \begin{pmatrix} z_2 \\ \overline{z}_1 \end{pmatrix}.$$

- $C^\infty(S^3, S^+)$: the set of smooth even spinors on S^3.

$$C^\infty(S^3, S^+) \ni \phi = \begin{pmatrix} u \\ v \end{pmatrix} \longleftrightarrow u + jv \in S^3\mathbf{H}\,.$$

Since Δ^+ is the spinor representation of the Clifford algebra $\mathrm{Clif}_2^c \simeq \mathbf{H}$: $End(\Delta_2^+) \simeq \mathrm{Clif}_2^c$, we have the $\mathbf{C}$-linear isomorphism $\mathbf{H} \xrightarrow{\sim} \Delta = \mathbf{C}^2$:

$$\Delta^+ \ni \begin{pmatrix} u \\ v \end{pmatrix} \longleftrightarrow u + jv \in \mathbf{H}\,.$$

So we may look an even spinor as a $\mathbf{H}$-valued function.

- On $C^\infty(S^3, S^+)$ we define the $\mathbf{R}$-Lie algebra structure by

$$\left[\,\phi_1\,,\,\phi_2\,\right] = \begin{pmatrix} v_1\bar{v}_2 - \bar{v}_1 v_2 \\ (u_2 - \bar{u}_2)v_1 - (u_1 - \bar{u}_1)v_2 \end{pmatrix},$$

for $\phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$.

This is equivalent to

$$\left[\,u_1 + jv_1,\, u_2 + jv_2\,\right] \;=\; (v_1\bar{v}_2 - \bar{v}_1 v_2) + j\,(\,(u_2 - \bar{u}_2)v_1 - (u_1 - \bar{u}_1)v_2\,)\,.$$

The trace of a spinor $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ is by definition

$$tr\,\phi \;=\; 2Re.u \;=\; u + \overline{u}.$$

$$tr\,[\phi, \psi] = 0$$

- The correspondence

$$C^\infty(M, \Delta^+) \simeq C^\infty(M, \mathbf{H}) \simeq C^\infty(M, \mathfrak{mj}(2, \mathbf{C}))$$

is convinient for the calculation.
Where $\mathfrak{mj}(2, \mathbf{C}) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} ; a, b \in \mathbf{C} \right\}$.
By it we identify $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ with $\begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}$. Hence the product $\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ of two spinors is given by the matrix multiplication of corresponding elements of $\mathfrak{mj}(2, \mathbf{C})$, and $trace\, \phi = u + \overline{u}$ follows immediatey.

- **Laurent polynomial type harmonic spinors and the residues**

  If φ is a harmonic spinor; $D\varphi = 0$, on $\mathbf{C}^2 \setminus \{0\}$ then we have the expansion

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z), \tag{6.1}$$

  uniformly convergent on any compact subset of $\mathbf{C}^2 \setminus \{0\}$.

  The coefficients $C_{\pm(m,l,k)}$ are given by the formula:

$$C_{\pm(m,l,k)} = \frac{1}{2\pi^2}\int_{S^3} \langle \varphi, \phi^{\pm(m,l,k)} \rangle \, d\sigma, \tag{6.2}$$

  where $\langle\, , \,\rangle$ is the inner product of S^+.

-

Lemma 6.2.

$$\begin{aligned} \int_{S^3} tr\,\varphi\, d\sigma &= 4\pi^2 Re.C_{+(0,0,1)}, \qquad (6.3)\\ \int_{S^3} tr\, J\varphi\, d\sigma &= 4\pi^2 Re.C_{+(0,0,0)}. \end{aligned}$$

The formulas follow from (6.2) if we take $\phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$,

Definition 6.3. 1. We call the series (6.1) *a spinor of Laurent polynomial type* if only finitely many coefficients $C_{\pm(m,l,k)}$ are non-zero . The space of spinors of Laurent polynomial type is denoted by $\mathbf{C}[\phi^{\pm}]$.

2. For a spinor of Laurent polynomial type φ we call the vector

$$\operatorname{res}\varphi = \begin{pmatrix} -C_{-(0,0,1)} \\ C_{-(0,0,0)} \end{pmatrix} \text{ the residue at 0 of } \varphi.$$

- We have the residue formula:

$$\operatorname{res}\varphi = \frac{1}{2\pi^2}\int_{S^3} \gamma_+(z)\varphi(z)\sigma(dz). \tag{6.4}$$

7 Extensions of the current algebras on S^3

- **Extensions of the current algebras on S^3**
- Basic vector fields and basic 1-form on S^3.

$$\begin{aligned}
e_+ &= -z_2\frac{\partial}{\partial\bar{z}_1} + z_1\frac{\partial}{\partial\bar{z}_2}, \qquad e_- = -\bar{z}_2\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial z_2} \\
\theta &= \left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial\bar{z}_1} - \bar{z}_2\frac{\partial}{\partial\bar{z}_2}\right).
\end{aligned}$$

the commutation relations;

$$[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta.$$

Dual basis :

$$\begin{cases}
\theta_0^* &= \frac{1}{2|z|^2}(\overline{z}_1 dz_1 + \overline{z}_2 dz_2 - z_1 d\overline{z}_1 - z_2 d\overline{z}_2), \\
\theta_1^* &= \frac{1}{2}(e_+^* + e_-^*) \\
\theta_2^* &= \frac{\sqrt{-1}}{2}(e_+^* - e_-^*)
\end{cases}$$

where

$$e_+^* = \frac{1}{|z|^2}(-\overline{z}_2 d\overline{z}_1 + \overline{z}_1 d\overline{z}_2), \quad e_-^* = \frac{1}{|z|^2}(-z_2 dz_1 + z_1 dz_2),$$

1.

$$\theta_j^*(\theta_k) = \delta_{jk}$$

2. very important relations: $\overline{\theta}_k = \theta_k$.

3. Integrable condition:

$$\begin{aligned}
\frac{\sqrt{-1}}{2} d\theta_0^* &= \theta_1^* \wedge \theta_2^* \,, \\
\frac{\sqrt{-1}}{2} d\theta_1^* &= \theta_2^* \wedge \theta_0^* \,, \\
\frac{\sqrt{-1}}{2} d\theta_2^* &= \theta_0^* \wedge \theta_1^* \,,
\end{aligned}$$

4. $\theta_0^* \wedge \theta_1^* \wedge \theta_2^* = d\sigma_{S^3}$: volume form.

5.

$$\int_{S^3} \theta_k f \, d\sigma \; = \; 0 \,, \qquad k = 0, 1, 2$$

for any function f on S^3.

• **2-cocycles on** $S^3\mathbf{H} = C^\infty(S^3, \mathbf{H}) \simeq C^\infty(S^3, S^+)$

$\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in S^3\mathbf{H}$,

Put

$$\Theta_k\,\varphi \;=\; \begin{pmatrix} \frac{1}{2}\theta_k & 0 \\ 0 & \frac{1}{2}\theta_k \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} \theta_k\,u \\ \theta_k\,v \end{pmatrix}, \qquad k = 0, 1, 2.$$

• For ϕ_1 and $\phi_2 \in S^3\mathbf{H}$, we put

$$c_k(\phi_1, \phi_2) \;=\; \frac{1}{2\pi^2}\int_{S^3} tr\,(\,\Theta_k\phi_1 \cdot \phi_2\,)\,d\sigma. \tag{7.1}$$

• For each $k = 0, 1, 2$, c_k defines a non-trivial 2-cocycle on the algebra $S^3\mathbf{H}$.

That is, c_k satisfies the equations:

$$c_k(\phi_1\,\phi_2) \;=\; -\,c_k(\phi_2,\,\phi_1) \tag{7.2}$$

$$c_k(\phi_1 \cdot \phi_2\,,\,\phi_3) + c_k(\phi_2 \cdot \phi_3\,,\,\phi_1\,) + c_k(\phi_3 \cdot \phi_1\,,\,\phi_2\,) = 0 \tag{7.3}$$

for any ϕ_1, ϕ_2, $\phi_3 \in S^3\mathbf{H}$,

and

$\nexists$ 1-cochain b such that $c_k(\phi_1.\phi_2) = b(\,[\phi_1, \phi_2]\,)$.

- **2-cocycles on the Laurent polynomial type harmonic spinors**

  $\mathcal{L} =$ Laurent polynomial type harmonic spinors on S^3

  $\mathcal{L}$ being a subalgebra of $S^3S^3\mathbf{H}$, c_k, for each $k = 0,1,2$, defines a non-trivial 2-cocycle on $\mathcal{L}$.

- **Central extension of $S^3\,\mathfrak{gl}(n,\mathbf{H})$**

  $\mathbf{C}$-valued 2-cocycles on the real Lie algebra $S^3\mathfrak{gl}(n,\mathbf{H}) = S^3\mathbf{H}\otimes\mathfrak{gl}(n,\mathbf{C})$:

  Extend the 2-cocycles c_k, $k = 0,1,2$ on $S^3\mathbf{H}$
  to $S^3\mathfrak{gl}(n,\mathbf{H})$ by

$$c_k(\,\phi_1\otimes X\,,\,\phi_2\otimes Y\,) = (X|Y)\,c_k(\phi_1,\phi_2)\,,\quad k = 0,1,2. \tag{7.4}$$

  where $(X|Y) = Trace(XY)$

  The 2-cocycle property follows from the fact

$$(XY|Z) = (YZ|X)$$

- Let a_k, $k = 0,1,2$, be three indefinite numbers.

  For each $k = 0,1,2$

  there is a central extension of the Lie algebra $S^3\mathfrak{gl}(n,\mathbf{H})$ by the 1-dimensional center $\mathbf{C}a_k$ associated to the cocycle c_k.

Theorem 7.1.

$$S^3\mathfrak{gl}(n,\mathbf{H})(a) = (\,S^3\mathbf{H}\otimes\mathfrak{gl}(n,\mathbf{C})\,)\oplus(\oplus_{k=0,1,2}\mathbf{C}a_k), \tag{7.5}$$

endowed with the following bracket becomes a real Lie algebra.

$$
\begin{aligned}
[\,\phi \otimes X\,,\,\psi \otimes Y\,]\hat{}\; &= \; (\phi \cdot \psi) \otimes XY \; - (\psi \cdot \phi) \otimes YX \\
&\quad + (X|Y) \sum_{k=0}^{2} c_k(\phi, \psi)\, a_k\,, \\
[\,a_k\,,\,\phi \otimes X\,]\hat{}\; &= \; 0\,,\; k = 0, 1, 2
\end{aligned}
$$

for ϕ, $\psi \, \in S^3\mathbf{H}$ *and any bases* X, $Y \in \mathfrak{gl}(n, \mathbf{H})$.

Check : $\mathbf{R}$-linear, antisymmetric, Jacobi identity.

- As a Lie subalgebra of $S^3\mathfrak{gl}(n,\mathbf{H})$ we have the Lie algebra of $\mathfrak{gl}(n,\mathbf{C})$-valued Laurent polynomial spinors on S^3 :

$$L\,\mathfrak{gl} \;=\; \mathbf{C}[\phi^{\pm}] \otimes_{\mathbf{C}} \mathfrak{gl}(n,\mathbf{C}).$$

The basis of $\mathbf{C}[\phi^{\pm}] \otimes \mathfrak{gl}(n,\mathbf{C})$:

$$\phi^{\pm(m,l,k)} \otimes E_{ij}\,, \qquad \begin{array}{c} 1 \leq i,\, j \leq n, \\ 0 \leq m,\, 0 \leq l \leq m,\, 0 \leq k \leq m+1 \end{array} .$$

As a Lie subalgebra of $S^3\mathfrak{gl}(n,\mathbf{H})$, $L\,\mathfrak{gl}$ has the central extension by the 2-cocycles $\tilde{c}_k$, $k = 0, 1, 2$, as well:

$$L\,\mathfrak{gl}(a) = L\,\mathfrak{gl} \,\oplus\, (\oplus_{k=0}^{2}\,\mathbf{C}a_k).$$

- The radial vector field d_0 on $\mathbf{C}^2 \setminus 0$ is extended to act on $L\,\mathfrak{gl}(a)$ as an outer derivation.

  Then, adjoining the derivation d, we have the second extension:

$$\widehat{\mathfrak{gl}} = \mathbf{C}[\phi^{\pm}] \otimes_{\mathbf{C}} \mathfrak{gl}(n, \mathbf{C}) \oplus (\oplus_{k=0}^{2} \mathbf{C} a_k) \oplus \mathbf{C} d.$$

8 Algebra of infinitesimal automorphisms on S^3 and its central extension

- Basis of vector fields on $S^3 \simeq \{|z|=1\} \subset \mathbf{C}^2$.

$$
\begin{aligned}
e_+ &= -z_2\frac{\partial}{\partial \bar{z}_1} + z_1\frac{\partial}{\partial \bar{z}_2}\,, \qquad e_- = -\bar{z}_2\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial z_2} \\
\theta &= \left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2} \right)
\end{aligned}
$$

with the commutation relations;

$$
[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta.
$$

- we define the polynomials:

$$
\begin{aligned}
v^k_{(l,m-l)} &= (e_-)^k z_1^l z_2^{m-l}. \\
m &= 0, 1, 2, \cdots \qquad l,\ k = 0, 1, \cdots, m
\end{aligned}
$$

Then $v^k_{(l,m-l)}$ is a harmonic polynomial on $\mathbf{C}^2 \setminus \{0\}$ restricted to $S^3 \subset \mathbf{C}^2 \setminus \{0\}$;

$$
\begin{aligned}
e_+ v^k_{(l,m-l)} &= -k(m-k+1)v^{k-1}_{(l,m-l)}, \\
e_- v^k_{(l,m-l)} &= v^{k+1}_{(l,m-l)}, \\
\theta v^k_{(l,m-l)} &= (m-2k)v^k_{(l,m-l)}\,.
\end{aligned}
$$

8.1 Witt algebra on S^3

- We shall investigate the space of vector fields on S^3 with *harmonic polynomial coefficients.*

Definition 8.1. $\mathbf{V}(S^3)$ is the set of vector fields on S^3 with harmonic polynomial coefficients.

Basis of $\mathbf{V}(S^3)$:

$$\begin{aligned} L^k_{(l,m-l)} &= v^k_{(l,m-l)}\,\theta_0\,, \\ E^k_{(l,m-l)} &= v^k_{(l,m-l)}\,e_+\,, \qquad 0 \le m, \quad 0 \le k, l \le m \\ F^k_{(l,m-l)} &= v^k_{(l,m-l)}\,e_-\,. \end{aligned}$$

$\mathbf{V}(S^3)$ is a Lie algebra with basis

$$\{L^k_{(l,m-l)},\, E^k_{(l,m-l)},\, F^k_{(l,m-l)}\,, \quad 0 \le m, \quad 0 \le k, l \le m\}$$

notation

$$\alpha = (\alpha_1, \alpha_2), \quad \beta = (\beta_1, \beta_2), \cdots, \quad \mathbf{1} = (1, 1), \quad k \cdot \mathbf{1} = (k, k)$$

$$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2)$$

$$|\alpha| = \alpha_1 + \alpha_2\,, \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

- The structure constants of $\mathbf{V}(S^3)$ are:

$$
\begin{aligned}
[\,L^h_\alpha\,,\,L^k_\beta\,] &= \sqrt{-1}(k-h+\frac{|\alpha|-|\beta|}{2})\sum_{j=0}^{h+k} C_j(\alpha,h;\beta,k)\,L^{h+k-j}_{\alpha+\beta-j\mathbf{1}} \\
[\,L^h_\alpha\,,\,E^k_\beta\,] &= \sqrt{-1}(k-\frac{1}{2}|\beta|+1)\sum_{j=0}^{h+k} C_j(\alpha,h;\beta,k)E^{h+k-j}_{\alpha+\beta-j\mathbf{1}} \\
&\quad -\sum_{j=0}^{h+k+1} C_j(\alpha,h+1;\beta,k)L^{h+k+1-j}_{\alpha+\beta-j\mathbf{1}} \\
[\,L^h_\alpha\,,\,F^k_\beta\,] &= \sqrt{-1}(k-\frac{1}{2}|\beta|-1)\sum_{j=0}^{h+k} C_j(\alpha,h;\beta,k)F^{h+k-j}_{\alpha+\beta-j\mathbf{1}} \\
&\quad -h(|\alpha|-h+1)\sum_{j=0}^{h+k-1} C_j(\alpha,h-1;\beta,k)L^{h+k-1-j}_{\alpha+\beta-j\mathbf{1}} \\
[\,E^h_\alpha\,,\,E^k_\beta\,] &= \sum_{j=0}^{h+k}\left(\,C_j(\alpha,h;\beta,k+1)-C_j(\alpha,h+1;\beta,k)\right)\,E^{h+k+1-j}_{\alpha+\beta-j\mathbf{1}} \\
[\,F^h_\alpha\,,\,F^k_\beta\,] &= \sum_{j=0}^{h+k-1}\{\,h(|\alpha|-h+1)(\,C_j(\alpha,h-1;\beta,k) \\
&\quad -k(|\beta|-k+1)C_j(\alpha,h;\beta,k-1)\}\,F^{h+k-j-1}_{\alpha+\beta-j\mathbf{1}} \\
[\,E^h_\alpha\,,\,F^k_\beta\,] &= \sum_{j=0}^{h+k+1} C_j(\alpha,h;\beta,k+1)F^{h+k+1-j}_{\alpha+\beta-j\mathbf{1}} \\
&\quad +h(|\alpha|-h+1)\sum_{j=0}^{h+k-1} C_j(\alpha,h-1;\beta,k)E^{h+k-1-j}_{\alpha+\beta-j\mathbf{1}} \\
&\quad -2\sum_{j=0}^{h+k} C_j(\alpha,h;\beta,k)L^{h+k-j}_{\alpha+\beta-j\mathbf{1}}
\end{aligned}
$$

In the above the constants $C_j(\alpha, h; \beta, k)$ are defined by the following:

Lemma 8.2.

1. *Given* α , h, β , k,

$$\exists\, C_j = C_j(\alpha, h; \beta, k) \equiv C_j(\begin{matrix} h \\ \alpha \end{matrix} ; \begin{matrix} k \\ \beta \end{matrix}), \quad j = 0, \cdots, h+k$$

that satisfy the relation

$$v_\alpha^h \cdot v_\beta^k = \sum_{j=0}^{h+k} C_j \, |z|^{2j} \, v_{\alpha+\beta-j\mathbf{1}}^{h+k-j}$$

2.

$$C_j(\begin{matrix} h \\ \alpha \end{matrix} ; \begin{matrix} k \\ \beta \end{matrix}) = C_j(\begin{matrix} k \\ \beta \end{matrix} ; \begin{matrix} h \\ \alpha \end{matrix})$$

3. *If* $\quad j_1 + j_2 = k_1 + k_2$,

$$C_{j_1}(\begin{matrix} p_1 \\ \alpha_1 \end{matrix} ; \begin{matrix} p_2 \\ \alpha_2 \end{matrix}) \, C_{j_2}(\begin{matrix} p_1 + p_2 - j_1 \\ \alpha_1 + \alpha_2 - j_1\mathbf{1} \end{matrix} ; \begin{matrix} p_3 \\ \alpha_3 \end{matrix})$$

$$= C_{k_1}(\begin{matrix} p_2 \\ \alpha_2 \end{matrix} ; \begin{matrix} p_3 \\ \alpha_3 \end{matrix}) \, C_{k_2}(\begin{matrix} p_2 + p_3 - k_1 \\ \alpha_2 + \alpha_3 - k_1\mathbf{1} \end{matrix} ; \begin{matrix} p_1 \\ \alpha_1 \end{matrix})$$

$$\Longleftarrow \qquad \left(v_{\alpha_1}^{p_1} v_{\alpha_2}^{p_2} \right) v_{\alpha_3}^{p_3} = v_{\alpha_1}^{p_1} \left(v_{\alpha_2}^{p_2} v_{\alpha_3}^{p_3} \right)$$

example

$$\begin{aligned} C_0(\alpha, 0;\, \beta, 0) &= 1, \\ C_m(m, l, k; m-l, l, m-k) &= (-1)^{m-l+k} \frac{l!(m-l)!}{m+1} \end{aligned}$$

$\psi DS \Longrightarrow$ Functions on the cotangent bundle $T^*M^n \setminus \{0\}$

$$CL(M^n) = \left\{ a = \sum_{-\infty<k\leq d} a_k(x,\xi) \right\}, \quad 0 \neq \xi \in T^*_x M\,,$$

: the ring of formal pseudo-differential symbols on M^n.

Here $a_k(x,\xi)$ are functions on the cotangent bundle with zero section removed *of homogeneous degree* k *in* ξ.

The multiplication in $CL(M^n)$ is:

$$a \cdot b = \sum_{\alpha} \frac{1}{\alpha!} \left(\partial_\xi^\alpha \, a \right) \left(\partial_x^\alpha \, b \right) . \tag{8.1}$$

where α denotes a multiindice

for example

For $M = S^1$ (8.1) coresponds to (**??**) by $\partial \longrightarrow \xi$:

$$[\partial^\alpha, \, f\partial^m] \longrightarrow \sum_{j=1} \begin{pmatrix} \alpha \\ j \end{pmatrix} f^{(j)}\xi^{m+\alpha-j} = \xi^\alpha \cdot f\xi^m = [\, \xi^\alpha, \, f\xi^m \,]$$

(remark: $(f\xi^m) \cdot \xi^\alpha = 0!$)

$$\psi DO \ni \sum_{-\infty<k\leq d} a_k(x)\partial^k \longrightarrow \sum_{-\infty<k\leq d} a_k(x)\xi^k \in CL(S^1)$$

- the non-commutative residue (Wodzcki):

$$Res\, a = \int_{\{|\xi|=1\}} a_{-n}(x,\xi)\, \alpha \wedge \omega^{n-1},$$

for $a = \sum_{-\infty<k\leq d} a_k(x,\xi) \in CL(M^n)$. a n-form on M.

$$tr\, a \;=\; \int_M Res\, a\, dx\,.$$

$$\alpha = \sum \xi_i dx_i\,;\ \text{ canonical 1-form,} \qquad \omega = d\alpha.$$

$\alpha \wedge \omega^{n-1}$: volume form on the cotangent sphere $S^*M = \{|\xi| = 1\}$

S : an ellptic differential operator of order m on M
with the leading symbol $s_m(x,\xi) > 0$, $\xi \neq 0$.

Theorem [Radul]

$$c(a,b) = \int_M Res(\,[\ln s_m, a] \cdot\, b\,)\,, \quad a, b \in CL(M) \tag{8.2}$$

gives a cocycle on the algebra $CL(M)$.

Note: though $\ln s_m(x,\xi) \notin CL(M)$, we have

$$[\ln s_m(x,\xi), CL(M)] \;\subset CL(M)\,.$$

There exist a central extension of $CL(M)$ by Radul cocycle c,

and a central extension of the subalgebra $Vect(M) \subset CL(M)$.

9 Central extensions of $Vect(S^3)$

Apply

" Theorem [Radul]

$$c(a,b) = \int_M Res(\, a \;\cdot\; [\,\ln s_m,\, b\,]\,)\,, \quad a,b \in CL(M) \tag{9.1}$$

gives a cocycle on the algebra $CL(M)$. "

to

$$\begin{aligned} Vect(S^3) \quad & \ni f_1(z)\frac{\partial}{\partial z_1} + f_2(z)\frac{\partial}{\partial z_2} + g_1(z)\frac{\partial}{\partial \overline{z}_1} + g_2(z)\frac{\partial}{\partial \overline{z}_2} \\ & \longrightarrow f_1(z)\xi_1 + f_2(z)\xi_2 + g_1(z)\overline{\xi}_1 + g_2(z)\overline{\xi}_2 \quad \in \quad CL(S^3) \end{aligned}$$

where

$$symb\,\frac{\partial}{\partial z_1} = \xi_1 \in T^*_z\,\mathbf{C}^2|S^3$$

etc.

Res is considered on $\mathbf{C}^2 \setminus 0$.

We take the elliptic symbol $s_m(z,\zeta) \;=\; symb\,(\Delta_{\mathbf{C}^2}) = |\zeta|^2$.

Definition 9.1. For $X, Y \in Vect(S^3)$, put

$$R(X,Y) = \int_{|\zeta|=1} (\, symb\, X \cdot \, [\, \ln|\zeta|^2 \, , \, symb\, Y \,]\,)\sigma(d\zeta)$$

where $|\zeta|^2 = symb.\Delta$.

$$c(X,Y) = \int_{|z|=1} R(X,Y)\sigma(dz), \quad i = 0, 1, 2$$

c gives a cocycle on $Vect(S^3)$.

9.1 Calculations of $R(X,Y)$

We shall calculate

$$R(X,Y) = \int_{|\zeta|=1} (\, symb\, X \cdot \, [\, \ln|\zeta|^2 \,,\, symb\, Y \,]\,) \sigma(d\zeta) \tag{9.2}$$

for

$$X \,, = \, \theta_0, \, e_+ \,, \, e_- \,, \tag{9.3}$$

and

$$Y \; = h(z) e_+ \,, \, h(z) e_- \,, \, h(z) \theta_0 \,.$$

where h is a harmonic function on $\mathbf{C}^2$ restricted to S^3.
After we shall take $h = v^k_{(m,l)}$, so that

$$X, Y = L^k_\alpha \,, \, E^k_\alpha \,, \, F^k_\alpha \,.$$

1.

$$R(h'\theta_0,\, he_+) = \frac{1}{3}\, h'(\theta e_+ - e_+)h \tag{9.4}$$

2.

$$R(h'e_+,\, h\theta_0) = \; h'(\frac{1}{3}e_+\theta + \frac{1}{2}e_+)h \tag{9.5}$$

3.

$$R(h'\theta_0,\, h\theta_0) = -\frac{1}{3}h'(\theta\theta + \frac{\partial}{\partial n})h \tag{9.6}$$

4.

$$R(h'\theta_0,\, he_-) = \frac{1}{3}h'\theta_0 e_- h \tag{9.7}$$

5.

$$R(h'e_- \; h\theta_0) = \frac{1}{3}h'(e_-\theta_0 - e_-)h \tag{9.8}$$

6.

$$R(\, h'e_+,\, he_-\,) = \frac{1}{3}\, h' \; (\, e_+ \cdot e_- \; + \; \nu\,)\, h \tag{9.9}$$

7.

$$R(h'e_-,\, he_+) = \; \frac{1}{3}h' \; (e_+ e_- + \nu)\, h = \frac{1}{3}\, h' \; (\, e_- \cdot e_+ \; + \; \overline{\nu}\,)\, h \tag{9.10}$$

8.

$$R(\, h'e_-,\, he_-\,) \; = \; \frac{1}{3}\, h' \; (\, e_- \cdot e_-\,)\, h\,. \tag{9.11}$$

9.

$$R(h'e_+, he_+) = \frac{1}{3}h'\,(\, e_+ \cdot e_+\,)\, h \tag{9.12}$$