Quantization

classical mechanics (Poisson algebra, $\cdot$, $\{\ \}$)

quantum mechanics (operator algebra, $*_\hbar$)

satisfying correspondence principle

$$\frac{1}{\sqrt{-1}\hbar}[\ ,\ ]_{*_\hbar} \xrightarrow[(\hbar\to 0)]{} \{\ ,\ \}$$

Poisson alg. $\rightsquigarrow$ Operator alg. $\left(\begin{array}{c}\text{non-comm.}\\ \text{assoc. alg.}\end{array}\right)$

$$f \longmapsto P_f^{(1/\hbar)}$$

s.t. $\displaystyle\frac{1}{\sqrt{-1}\hbar}[P_f^{(1/\hbar)}, P_g^{(1/\hbar)}] \xrightarrow[(\hbar\to 0)]{} \{f, g\}$

$*$ non-comm. assoc. alg. $\xrightarrow[\hbar\to 0]{}$ comm. assoc. alg.

$\downarrow$

Space = spec

Kohn-Nirenberg quantization

$(\mathbb{Q}, \mathbb{P}) = (q_1, \dots, q_n, p_1, \dots, p_n)$

$\cdots$ Darboux coord.

$\{q_i, p_j\} = \delta_{ij} \rightsquigarrow \frac{1}{\sqrt{-1}\hbar}[\hat{q}_i, \hat{p}_j] = \delta_{ij} \cdots (H)$

$\hat{q}_i : f(\mathbb{Q}) \mapsto q_i \cdot f(\mathbb{Q}) \cdots$ multiplication op.

$\hat{p}_j : f(\mathbb{Q}) \mapsto \frac{\hbar}{\sqrt{-1}} \partial_{q_j} f(\mathbb{Q}) \cdots$ differential op.

REM. Problem of domains of these operators is delicate.

$$\begin{array}{ccc} \iota : \text{"Func"} & \longrightarrow & A := \langle \hat{\mathbb{Q}}, \hat{\mathbb{P}} \rangle / (H) \\ \varphi & \longrightarrow & \iota(\varphi) \end{array}$$

$$\varphi * \psi := \iota^{-1}(\iota(\varphi) \circ \iota(\psi))$$

Roughly speaking, this proceedure gives non-commutative assoc. product $*$ on "Func."

Precisely, appropriate symbol class

$$\iota_{KN} : \text{Func.} \longrightarrow \mathcal{A} := \Psi DO.$$

$$p^\alpha q^\beta \longmapsto \hat{q}^\beta \hat{p}^\alpha$$

Fourier transf.

$$\frac{\hbar}{\sqrt{-1}} \partial_q f(q) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} p \cdot \hat{f}(p)$$

$$\iota_{KN}(\varphi(p, q)) f(q)$$

$$:= \int \varphi(q, p)\, e^{\frac{\sqrt{-1}}{\hbar}(q-q')\cdot p} f(q')\, đq'\, dp$$

$$\left(đ := \frac{1}{\sqrt{2\pi\hbar}^{\,n}}\, d \right)$$

$P_\varphi := \iota_{KN}(\varphi(p, q))$ ... pseudo-differential op. with symbol φ.

TH (symbol calculus)

$$P_\varphi \circ P_\psi = P(\varphi *_{KN} \psi),$$

where

$$\varphi *_{KN} \psi = \sum_{\alpha:\text{multi index}} \frac{(-\sqrt{-1}\hbar)^{|\alpha|}}{\alpha!} \partial_p^\alpha \varphi(p, q) \cdot \partial_q^\alpha \psi(p, q).$$

Furtheremore, we have

$$\text{symbol of } \frac{1}{\sqrt{-1}\hbar}[P_\varphi, P_\psi] \xrightarrow[(\hbar \to 0)]{} \{\varphi, \psi\}.$$

Proof.

$$
\begin{aligned}
&((P_\varphi \circ P_\psi) f)(q) \\
&= \int \varphi(q,p)\, e^{\frac{i}{\hbar}(q-q')\cdot p} \\
&\qquad \times \left(\int \psi(q',p')\, e^{\frac{i}{\hbar}(q'-q'')p'} f(q'')\, dq''\, dp' \right) dq'\, dp \\
&= \iint \varphi(q,p)\, \psi(q',p') \times e^{\frac{i}{\hbar}\{(q-q'')p' + (q'-q)(p'-p)\}}\, dq'\, dp \\
&\qquad \times f(q'')\, dq''\, dp' \\
&\overset{\text{Taylor}}{=} \iint \left(\sum_\alpha \frac{(p-p')^\alpha}{\alpha!} \partial_{p'}^\alpha \varphi(q,p') \right) \times \psi(q',p')\, e^{\frac{i}{\hbar}(q'-q)(p'-p)}\, dq'\, dp \\
&\qquad \times e^{\frac{i}{\hbar}(q-q'')p'} f(q'')\, dq''\, dp' \\
&= \iint \left(\sum_\alpha \frac{1}{\alpha!} \partial_{p'}^\alpha \varphi(q,p')\, \psi(q',p') \cdot \left(-\frac{\hbar}{i}\partial_{q'}\right)^\alpha e^{\frac{i}{\hbar}(q'-q)(p'-p)} \right) dq'\, dp \\
&\qquad \times e^{\frac{i}{\hbar}(q-q'')p'} f(q'')\, dq''\, dp' \\
&\overset{\substack{\text{integral} \\ \text{by} \\ \text{parts}}}{=} \iint \left(\sum_\alpha \frac{\left(\frac{\hbar}{i}\right)^{|\alpha|}}{\alpha!} \partial_{p'}^\alpha \varphi(q,p') \cdot \partial_{q'}^\alpha \psi(q',p') \cdot e^{\frac{i}{\hbar}(q'-q)(p'-p)} \right) dq'\, dp \\
&\qquad \times e^{\frac{i}{\hbar}(q-q'')p'} f(q'')\, dq''\, dp' \\
&\overset{\substack{\text{Fourier} \\ \text{inversion} \\ \text{formula}}}{=} \int \left(\sum_\alpha \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \partial_{p'}^\alpha \varphi(q,p')\, \partial_q^\alpha \psi(q,p') \right) \\
&\qquad \times e^{\frac{i}{\hbar}(q-q'')p'} f(q'')\, dq''\, dp' \\
&= (P_{\varphi *_{KN} \psi} f)(q)
\end{aligned}
$$

REM. $\int$ should be replaced with os-$\int$

Thus, the calculus above is formal.

In general, P_φ is not Hermitian.

Weyl quantization

$$\iota_W : \text{"Func"} \longrightarrow \mathcal{A}$$
$$\varphi \longmapsto W_\varphi$$

where

$$W_\varphi := \int \hat{\varphi}(\eta, \xi)\, e^{i(\eta \hat{q} + \xi \hat{p})}\, d\eta\, d\xi$$

… Weyl type ΨDO.

Then we have

TH (Weyl calculus. Moyal product)

$$W_\varphi \circ W_\psi = W_{\varphi *_w \psi},$$

where

$$\varphi *_w \psi = \sum_{\alpha, \beta} \frac{\left(\frac{i\hbar}{2}\right)^{|\alpha+\beta|}}{\alpha! \beta!} \partial_p^\alpha \partial_q^\beta \varphi(p, q) \cdot \partial_q^\alpha (-\partial_p)^\beta \psi(q, p)$$

Furthermore,

$$\text{Symbol of } \frac{1}{\hbar\sqrt{-1}} [W_\varphi, W_\psi] \xrightarrow[\hbar \to 0]{} \{\varphi, \psi\}.$$

Proof

$$\mathcal{F} : \mathbb{R}^{2n}_{(q,p)} \longrightarrow \hat{\varphi} \in \mathbb{R}^{2n}_{(\eta,\xi)} \text{ : Fourier transf.}$$

$$P := \hat{p}\ ,\quad Q := \hat{q}$$

$$\iint \hat{\varphi}(\eta_1,\xi_1)\,\hat{\psi}(\eta_2,\xi_2)\,e^{i(\eta_1 Q+\xi_1 P)}\cdot e^{i(\eta_2 Q+\xi_2 P)}\,d\eta_1 d\xi_1 d\eta_2 d\xi_2$$

$$\overset{BCH}{=} \iint \hat{\varphi}(\eta_1,\xi_1)\,\hat{\psi}(\eta_2,\xi_2)\cdot e^{\underbrace{\frac{i\hbar}{2}(\xi_1\eta_2-\xi_2\eta_1)}_{\text{can be seen as group 2 cocycle}}}\cdot e^{i(\xi P+\eta Q)}\,d\eta_1 d\xi_1 d\eta_2 d\xi_2 \quad \begin{pmatrix}\xi=\xi_1+\xi_2\\ \eta=\eta_1+\eta_2\end{pmatrix}$$

$$\overset{Taylor}{=} \iint \hat{\varphi}(\eta_1,\xi_1)\,\hat{\psi}(\eta_2,\xi_2)\left(\sum_n \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n(\xi_1\eta_2-\xi_2\eta_1)^n\right) \times e^{i(\xi P+\eta Q)}\,d\eta_1 d\xi_1 d\eta_2 d\xi_2$$

$$= \iint \sum_{\alpha,\beta}\frac{\left(\frac{i\hbar}{2}\right)^{|\alpha+\beta|}}{\alpha!\,\beta!}\,\xi_1^{\alpha}\eta_1^{\beta}\hat{\varphi}(\eta_1,\xi_1)\cdot\eta_2^{\alpha}(-\xi_2)^{\beta}\hat{\psi}(\eta_2,\xi_2) \times e^{i(\xi P+\eta Q)}\,d\eta_1 d\xi_1 d\eta_2 d\xi_2$$

$$= \iint \sum_{\alpha,\beta}\frac{\left(\frac{i\hbar}{2}\right)^{|\alpha+\beta|}}{\alpha!\,\beta!}\,\widehat{(-i\partial_p)^{\alpha}(-i\partial_q)^{\beta}\varphi}(\eta_1,\xi_1)\cdot\widehat{(-i\partial_q)^{\alpha}(i\partial_p)^{\beta}\psi}(\eta-\eta_1,\xi-\xi_1) \times e^{i(\xi P+\eta Q)}\,d\eta_1 d\xi_1 d\eta\, d\xi$$

$$\overset{convolution}{=} \int \sum_{\alpha,\beta}\frac{\left(\frac{i\hbar}{2}\right)^{|\alpha+\beta|}}{\alpha!\,\beta!}\,\widehat{\left(\partial_q^{\beta}\partial_p^{\alpha}\varphi(q,p)\cdot\partial_p^{\beta}(-\partial_q)^{\alpha}\psi(q,p)\right)}(\eta,\xi) \times e^{i(\xi P+\eta Q)}\,d\eta\, d\xi$$

$$= \int \widehat{\varphi *_w \psi}(\eta,\xi)\,e^{i(\xi P+\eta Q)}\,d\eta\, d\xi$$

$$\therefore\ W_\varphi \circ W_\psi = W_{\varphi *_w \psi}\,.$$

Consider "formal" Weyl algebra bundle W_v on a local Darboux coordinates.
Glueing these algebra bundles, we would like to construct globally defined non-commutative associative algebra bundle which can be regard as deformation quantization in some sense.

Deformation quantization

- Background information
  of Quantization

(1) Existence (Deformation quantization)

$\forall (M, \omega)$: Symplectic manifold

$\exists\, \mathcal{A} := (C^\infty(M)[[\nu]], *)$

… Deformation quantization of (M, ω)

- DeWilde - Lecomte
- Omori - Maeda - Yoshioka
- Fedosov
- Kontsevich

(2) classification

$$\{ D.Q. \text{ on } (M, \omega) \} / \sim$$
$$\downarrow$$
$$\check{H}^2(M, \mathbb{R}[[\nu]])$$

- Deligne
- Omori - Maeda - Miyazaki - Yoshioka
- Nest - Tsygan
- Kontsevich

◎ Deformation quantization (def.)

DEF. (Bayen - Flato - Fronsdal - Lichnerowicz - Sternheimer)

A deformation quantization/star product of a Poisson manifold $(M, \{\ ,\ \})$ is a product $*$ on $C^\infty(M)[\![\nu]\!]$, the space of formal power series of parameter ν with coefficients in $C^\infty(M)$:

$$f * g = f \cdot g + \nu \pi_1(f, g) + \cdots + \nu^n \cdot \pi_n(f, g) + \cdots$$

satisfying

(1) $*$ is associative,

(2) $\pi_1(f, g) = \frac{1}{2\sqrt{-1}}\{f, g\}$

(3) each π_n is an $\mathbb{C}[\![\nu]\!]$-bilinear, bidifferential operator.

◎ Weyl manifold, contact Weyl manifold for deformation quantization

DEF. W : Weyl algebra

$\overset{\text{def}}{\Longleftrightarrow}$ binary operator : $*$

generators ;

$\nu,\ Z_1 = X_1, \dots, Z_n = X_n, Z_{n+1} = Y_1, \dots, Z_{2n} = Y_n$

relations :

$[X_i, X_j] = [Y_i, Y_j] = [\nu, X_i] = [\nu, Y_j] = 0$

$[X_i, Y_j] = -\nu\delta_{ij} \quad \left([Z_i, Z_j] = \nu\Lambda_{ij},\ \Lambda = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\right)$

where $[a, b] := a*b - b*a$.

$\exists\ \bar{\cdot} : a \mapsto \bar{a}\ ;\ \overline{a*b} = \bar{b}*\bar{a},\ \bar{\nu} = -\nu,\ \bar{Z}_i = Z_i$

$\exists\ d := \deg : \nu^{l} Z^{\alpha} \mapsto 2l + |\alpha|$, α: multi-index □

DEF. C : contact Weyl algebra

$\overset{\text{def}}{\Longleftrightarrow}$ $C = \tau\mathbb{C} + W$

where W is a Weyl algebra with

$[\tau, \nu] = 2\nu^2, \quad [\tau, Z_i] = \nu Z_i.$ □

◎ automorphism of algebra

DEF. Φ ; ν-automorphism of Weyl algebra W

$\overset{def}{\Longleftrightarrow}$ (1) Φ : $\mathbb{C}$-linear

(2) $\Phi(\nu) = \nu$

(3) $\Phi(a * b) = \Phi(a) * \Phi(b)$

(4) $\Phi(\bar{a}) = \overline{\Phi(a)}$. □

We consider the following algebra bundles ;

$$W_M := \coprod_U W_U / \sim \quad (W_U := U \times W)$$

$$C_M := \coprod_U C_U / \sim \quad (C_U := U \times C)$$

where "$\sim$" is a suitable equivalence relation discussed below :

DEF. Let $U_{(x,y)}$ be a Darboux chart of symplectic manifold M.

$$\# : C^\infty(U)[\![\nu]\!] \longrightarrow \Gamma(W_U) \subset \Gamma(C_U)$$

$$f \longmapsto f^\# = \sum_{\alpha,\beta} \frac{X^\alpha Y^\beta}{\alpha!\,\beta!} \partial_x^\alpha \partial_y^\beta f(x,y)$$

… (local) Weyl function

$$\mathcal{F}(W_U) := \{ f^\# \mid f \in C^\infty(U)[\![\nu]\!] \} .$$ □

DEF. Φ ; a "local" Weyl diffeo.

$\overset{\text{def}}{\Longleftrightarrow}$ (1) $\Phi_z : W_z \longrightarrow W_{\phi(z)}$ is a ν-auto.
where $z \in M$ (base mfd.), ϕ is the local diffeo. induced by Φ.
(2) $\Phi^*(\mathcal{F}(W_{\phi(V)})) = \mathcal{F}(W_V)$. □

PROP. Φ is a "local" Weyl diffeo.
$\Rightarrow$ $\phi : V \to \phi(V)$ is a symplectic diffeo. □

Conversely,

TH. ([OMY, 3.7]) For any symplectic diffeo. $\phi : V \to \phi(V)$, there exists a Weyl diffeo. Φ which induces ϕ. □

This theorem plays a crucial role to glue trivial algebra bundle together preserving Weyl functions $\{\mathcal{F}(V)\}_V$. (i.e. $\amalg W_V / \sim$)

DEF. $\Psi : C \longrightarrow C$, ν-automorphism

$\overset{\text{def.}}{\Longleftrightarrow}$ (1) Ψ ; algebra isomorphism,

(2) $\Psi|_W$; ν-auto. of Weyl algebra. $\square$

DEF. $\Psi ; C_V \to C_{\phi(V)}$

is a modified contact Weyl diffeo.

(MCWD for short.)

$\overset{\text{def.}}{\Longleftrightarrow}$ (1) Ψ_Z : ν- automorphism,

(2) $\Psi|_{W_V}$: Weyl diffeomorphism. $\square$

This notion is also important to glue trivial contact Weyl algebra bundle together.

Properties of MCWD

<u>DEF.</u> $\widetilde{\tau}_U(z) \overset{\text{def}}{:=} \tau + \sum_{i,j} z_i \omega^{ij} Z_j \in \Gamma(C_U)$

$$[\widetilde{\tau}_U(z), F] = 2\nu^2 \partial_\nu F + \nu \sum_i (z_i)^{\#} \cdot \partial_{z_i} F$$

<u>REM.</u> $\mathrm{ad}\left(\frac{1}{\nu}F\right)(a) := \begin{cases} \frac{1}{\nu}[F, a] & (a \in W) \\ 2F + \frac{1}{\nu}[F, \tau] & (a = \tau) \end{cases}$ $\square$

<u>TH.</u> ${}^\forall \Psi$: modified contact Weyl diffeo.

$\exists f \in C^\infty(U)[[\nu]]$, $\exists a \in (\nu^2) \in C^\infty(U)[[\nu]]$

; $\Psi^* \widetilde{\tau}_{\phi(U)} = \widetilde{\tau}_U + f^{\#} + \underline{a(\nu^2)}$ $\blacksquare$

<u>DEF.</u> A modified contact Weyl diffeo. Ψ satisfying

$\Psi^* \widetilde{\tau}_{\phi(U)} = \widehat{\tau}_U + f^{\#}$

is called a contact Weyl diffeomorphism.

(CWD, for short.) $\square$

<u>TH.</u> [OMY, 4.7]

${}^\forall \phi$: local symplectic diffeomorphism

$\exists_1\ \hat{\phi} := \Psi$: contact Weyl diffeomorphism

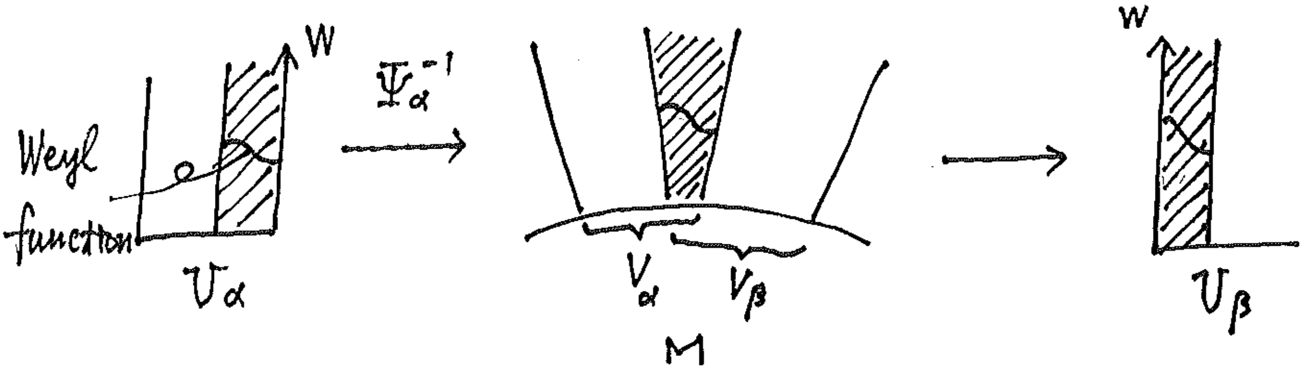
; Ψ induces ϕ. $\blacksquare$

DEF. (Weyl manifold)

$$\left\{ W_M \xrightarrow{\pi} M = \bigcup_\alpha V_\alpha \,,\ \Psi_\alpha : \pi^{-1}(V_\alpha) \to W_{V_\alpha} := V_\alpha \times W \right\}$$

is called a Weyl manifold.

$\overset{\text{def.}}{\Longleftrightarrow}$ ${}^\forall \Psi_{\alpha\beta} := \Psi_\beta \circ \Psi_\alpha^{-1} : W_{V_\alpha} \to W_{V_\beta}$ is a Weyl diffeo.



TH. [Omori - Maeda - Yoshioka]

${}^\forall(M, \omega)$: symplectic manifold

$\exists \{ W_M \xrightarrow{\pi} M \}$: Weyl manifold

; $M = \bigcup_\alpha V_\alpha$, V_α : Darboux chart. ∎

TH. [Omori - Maeda - Yoshioka - Miyazaki]

Assume that (M, ω) is a symplectic manifold.

Then,

$$\left\{ \{ W_M \xrightarrow{\pi} M : \text{Weyl manifold} \} \right/ \sim$$

$$\longleftrightarrow \{ * : (\text{hermite type) star product} \} / \sim$$

$$\longleftrightarrow [\omega] + \check{H}^2(M, \nu^2 \mathbb{R}[\![\nu^2]\!])$$ ∎

Automorphism.

We will treat

$$\mathrm{Aut}(C_M) := \{\Psi : C_M \to C_M \mid \text{global MCWD}\}.$$

Before it. we prepare fundamental Proposition.

PROP. [Y. Prop. 2.5 and 2.18]

(1) $\Phi : W \longrightarrow W$, ν-auto. of Weyl algebra

$\Rightarrow \exists_1 A = (a_{ij}) \in Sp(n. \mathbb{R})$, $\exists_1 F \in W_3^{+,0} = \{a \in W \mid a = \sum_{2l+|\alpha| \geq 3} a_{l\alpha} \nu^l Z^\alpha,\ \bar{a} = a\}$

s.t. $\Phi = \hat{A} \circ e^{\frac{1}{\nu} \mathrm{ad} F}$

where $\hat{A} Z^i = \sum a_{ij} Z^j$, $\hat{A}\nu = \nu$.

We call $\hat{A}$ a Weyl lift of A.

(2) $\Psi : C \longrightarrow C$, ν-auto. of contact Weyl algebra

$\Rightarrow \exists_1 A = (a_{ij}) \in Sp(n. \mathbb{R})$, $\exists_1 F \in W_3^{+,0}$, $\exists_1 c(\nu^2) \in \mathbb{R}[[\nu^2]]$

s.t. $\Psi = \hat{A} \circ e^{\mathrm{ad}\left(\frac{1}{\nu}(F + c(\nu^2))\right)}$ ∎

Proof.

(1) $\mathfrak{M} := \{ a \in \sum_{2\ell + |\alpha| \geqq 1} a_{\ell\alpha}\, \nu^{\ell} Z^{\alpha} \in W \}$ ... unique maximal ideal.

1st. $\Phi(\mathfrak{M}) \subset \mathfrak{M}$

$\therefore\ \Phi(Z^i) = \sum a_{ij} Z^j + O(2)$ ..... (i)

where

$O(2)$ is the collection of the terms degree $\geqq 2$.

Applying (i) to $[Z^i, Z^j] = \nu \Lambda^{ij}$, we see

$A \in Sp(n.\ \mathbb{R})$.

2nd $\hat{A}^{-1} \circ \Phi(Z^i) = Z^i + g^i_{(2)} + O(3)$ ..... (ii)

where

$g^i_{(2)}$ is the collection of the terms of homogeneous degree $= 2$.

$O(3)$ is the collection of the terms degree $\geqq 3$.

Applying (ii) to $[Z^i. Z^j] = \nu \Lambda^{ij}$, we have

$$\nu \partial_{i+n} g^j_{(2)} = \nu \Lambda^{i\ell} \partial_\ell g^j_{(2)} = [Z^i, g^j_{(2)}]$$

$$= [Z^j, g^i_{(2)}] = \nu \Lambda^{jk} \partial_k g^i_{(2)} = \nu \partial_{j+n} g^i_{(2)}$$

This means "closedness", i.e.

$$d(g^j_{(2)} dZ^{j+n} + g^i_{(2)} dZ^{i+n}) = 0$$

Then. by Poincaré's Lemma,

$\exists_1$ $F_{(3)}$: homogeneous degree $= 3$

$$\text{s.t. } \frac{1}{\nu}[Z^i, F_{(3)}] = g^{1\,i}_{(2)}$$

$$\therefore \hat{A}^{-1} \circ \Phi(Z^i) = e^{\frac{1}{\nu}F(3)}(Z^i) + O(3).$$

3rd Repeating this process, we obtain

$$\Phi(Z^i) = \hat{A} \circ \underbrace{e^{ad(\frac{1}{\nu}F_{(3)})} \circ e^{ad(\frac{1}{\nu}F_{(4)})} \circ \cdots \circ e^{ad(\frac{1}{\nu}F_{(k)})\cdots}(Z^i)}$$

$$k \to \infty \downarrow \text{ Baker-Campbell-Hausdorff.}$$

$$e^{ad(\frac{1}{\nu}F)} \quad (\exists F \in W_3^{0+})$$

$$\therefore \Phi = \hat{A} \circ e^{ad(\frac{1}{\nu}F)}$$

The uniqueness is inductively checked!

(2) By (1), we have $\Psi|_W = \hat{A} \circ e^{\mathrm{ad}(\frac{1}{\nu}F)} =: \psi$ on C $\cdots$ (iii)

$\therefore\ \psi^{-1} \circ \Psi(Z^i) = Z^i \cdots$ (iv)

Applying (iv) to $[\tau, Z^i] = \nu Z^i$.

$\psi^{-1} \circ \Psi(\tau) = \tau + b \quad (\exists\, b = b_0 + \nu^2 b_2 + \cdots + \nu^{2k} b_{2k} + \cdots \in \mathbb{R}[\![\nu^2]\!])$.

Put

$$c(\nu^2) := \sum \nu^{2k} \frac{b_{2k}}{2(1-2k)},$$

then we obtain

$$e^{\mathrm{ad}(\frac{1}{\nu}c(\nu^2))} Z^i = Z^i.$$

$$\therefore\ e^{\mathrm{ad}(-\frac{1}{\nu}c(\nu^2))} \circ \psi^{-1} \circ \Psi = 1 \text{ on } C.$$

$$\therefore\ \Psi \overset{\text{(iii)}}{=} A \circ e^{\mathrm{ad}(\frac{1}{\nu}(F + c(\nu^2)))}. \quad \blacksquare$$

Summerizing what mentioned above,

(i) Ψ ; ν-auto. of contact Weyl alg. C

$$\Rightarrow \Psi = \hat{A} \circ \exp\left(\frac{1}{\nu}(c(\nu^2) + F)\right)$$

$$(\exists A \in Sp(n.\mathbb{R}),\ \exists c(\nu^2) \in \mathbb{R}[\![\nu^2]\!].\ \exists F \in W)$$

(ii) Ψ : MCWD which induces the id. map on the base space

$$\Rightarrow \Psi = 1 \circ \exp\left(\frac{1}{\nu}(g(\nu^2) + \nu^2 f^{\#})\right)$$

$$(\exists g(\nu^2) \in C^\infty(M)[\![\nu^2]\!],\ f^{\#} \in \mathcal{F}_M)$$

(iii) Ψ : MCWD

$$\Rightarrow \Psi = \tilde{\psi} \circ \exp\left(\frac{1}{\nu}(g(\nu^2) + \nu^2 f^{\#})\right)$$

proof of (iii)

$$\Psi \xrightarrow[\text{induce a symp. diffeo. on } M.]{} \psi_M \xrightarrow[\text{canonical MCWD lift}]{} \tilde{\psi}_M$$

Then,

$$\tilde{\psi}_M \circ \Psi \in \underline{Aut}(C_M)$$

$$\overset{(ii)}{\Rightarrow} \tilde{\psi}^{-1} \circ \Psi = \exp\left(\frac{1}{\nu}(g(\nu^2) + \nu^2 f^{\#})\right)$$

◎ Automorphism (group)

<u>DEF.</u> $\mathrm{Aut}(C_M) := \left\{ \Psi : C_M \to C_M \;\middle|\; \begin{array}{l} \text{fiberwise } \nu\text{-auto} \\ \Psi^*(\mathcal{F}_M) = \mathcal{F}_M \end{array} \right\}$

$\underline{\mathrm{Aut}}(C_M) := \left\{ \Psi \in \mathrm{Aut}(C_M) \;\middle|\; \begin{array}{l} \Psi \text{ induces the} \\ \text{identity map} \\ \text{on the base space} \end{array} \right\}$

We would like to consider the following sequence;

$$1 \to \underline{\mathrm{Aut}}(C_M) \to \mathrm{Aut}(C_M) \to \mathrm{Diff}(M, \omega) \to 1 .$$

Before studying the above sequence, we would like to note several basics relating to infinite dimensional manifold and infinite dimensional Lie group.

Fact (Omori - de la Haap)

G Banach-Lie group

$\curvearrowright$ effective

M : finite dimensional smooth manifold

$\Rightarrow \dim G < \infty$. □

In order to treat $Aut(C_M)$ and $Diff(M, \omega)$, as Lie groups, we need a framework of infinite dimensional calculus, manifolds, and Lie groups beyond Banach-category!

$\Downarrow$

Difficulties (which we need to overcome)

(1) How can we construct infinite dimensional calculus?
What is smoothness?

(2) Definitions of manifold, (co) tangent space, Lie group, exponential.

As to (1) (cf. [KM] ⋯ Kriegl-Michor)

- smoothness of curves ⟶ no problem!
- smoothness of maps

$\Updownarrow$

"smooth curves" $\mapsto$ "smooth curves"

◎ convenient space E

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall c_1 \in C^\infty(\mathbb{R}, E)$, $\exists c_2 \in C^\infty(\mathbb{R}, E)$; $c_2' = c_1$

$\Longleftrightarrow$ $\forall c : \mathbb{R} \to E$ ($\forall \ell \in E^*$; $\ell \circ c$: smooth $\Longleftrightarrow$ c : smooth)

$\Longleftrightarrow$ $\forall B$: bdd. closed convex set $\subset E$; $E_B :=$ linear span of B is Banach with norm

$$p_B(v) := \inf\{\lambda > 0 ; v \in \lambda B\}.$$

◎ Everything else is a theorem!

- linearity, - Leibniz rule
- chain rule, - mean value
- Taylor, - the fundamental theorem
- cartesian closedness
  (i.e. $C^\infty(U, C^\infty(V, E)) \cong C^\infty(U \times V, E)$.)

◎ The definition of manifolds modeled on convenient spaces is given in the standard manner.

◎ Tangent vectors / bundles

We investigate tangent vectors seen as kinematic tangent vector (via curves), and then "push forward" is defined in the usual manner.

◎ Differential forms

There is only one suitable class satisfying all requirements (cf [KM, P351]).

$$\Omega^k(M) := C^\infty\big(\underbrace{L^k_{alt}}_{\text{alternative}}(TM,\ M\times\mathbb{R})\big)$$

$\Downarrow$

$\Omega^k(M)$ is closed under pull-back f^*, wedge product $\wedge$, exterior derivative d, interior product ι_X, Lie derivative L_X!

(2) Lie groups

The definition of "Lie group" is given in the usual manner.

◉ regular Lie groups

regularity ..... introduced by
- Omori-Maeda-Yoshioka-Kobayashi '82
- Milnor '84

$\downarrow$

"smooth curves in Lie algebras integrate to smooth curves in Lie groups in a smooth way"

◉ Precise definition

$$^{\forall} X \in C^\infty(\mathbb{R}, \mathfrak{g}) \quad (\mathfrak{g} := \mathrm{Lie}\, G)$$

$$\exists\, g \in C^\infty(\mathbb{R}, G)$$

$$\text{s.t. } \partial_t g(t) = R_{g(t)} X(t)$$

REM. $X(t) := tX \rightsquigarrow g(t) = \exp(tX)$.

TH. (i) $\mathrm{Aut}(C_M)$ is a Lie group modeled on

$$\mathcal{C} := C^\infty(M)[[\nu^2]] + \nu^2 \mathcal{F}_M .$$

(ii) $\mathrm{Aut}(C_M)$ is a Lie group modeled on

$\mathfrak{X}(M, \omega) \times \mathcal{C}$, where

$\mathfrak{X}(M, \omega)$ is the Lie algebra of all symplectic vector fields.

Combining the above theorem, with. TH. 4.7. in [OMY], we obtain,

Prop. $\forall \Psi \in \mathrm{Aut}(C_M)$

$\exists_1 \tilde{\psi}$: MCWD (canonical lift of ψ)

$\exists_1\ g(\nu^2) + \nu^2 f^{\#}(\nu^2)$

s.t. $\Psi = \tilde{\psi} \circ e^{\mathrm{ad}\left(\frac{1}{\nu}\left(g(\nu^2)+\nu^2 f^{\#}(\nu^2)\right)\right)}$.

Then we have

$$\mathrm{Aut}(C_M) \longleftrightarrow \mathrm{Diff}(M,\omega) \times C$$

$$\Psi \longleftrightarrow (\psi,\ g(\nu^2)+\nu^2 f^{\#}(\nu^2))$$

Proof.

$\Psi \rightsquigarrow \psi$: induced symp. diffeo. $\rightsquigarrow \tilde{\psi}$: lift

$\cdot\ \tilde{\psi}^{-1} \circ \Psi \in \mathrm{Aut}(C_M)$.

$\therefore\ \tilde{\psi}^{-1} \circ \Psi \overset{\text{TH.}}{=} e^{\mathrm{ad}\frac{1}{\nu}\left(g(\nu^2)+\nu^2 f^{\#}(\nu^2)\right)}$ ∎

Fundamental Theorem.

(1) $1 \to N \to G \to M \to 1$
is an exact sequence of Fréchet-Lie groups.

- $\exists M \supset U \xrightarrow{s} M$, local smooth section
- M, N ; regular

$\Rightarrow G$: regular $\square$

(2) Assume that M be compact. Then,
$\mathrm{Diff}(M, \omega) = \{ \varphi : \text{symplectic diffeo. on } M \}$
is a regular Fréchet-Lie group
modeled on

$$\mathfrak{X}(M, \omega) = \{ X : \text{symplectic vector field on } M. \}. \quad \square$$

Combining the above PROP. and Fundamental Theorem, we have.

TH. (1) $1 \to \underline{\mathrm{Aut}}(C_M) \to \mathrm{Aut}(C_M) \to \mathrm{Diff}(M, \omega) \to 1$

is an exact sequence of Fréchet-Lie groups, where (M, ω) is a compact symplectic manifold.

(2) $\underline{\mathrm{Aut}}(C_M)$ is regular.

(3) $\mathrm{Diff}(M, \omega)$ is regular.

(4) $\mathrm{Aut}(C_M)$ is regular.

Proof.

We first show $Aut(C_M)$ is a Fréchet-Lie group modeled on $\mathcal{X}(M,\omega) \times C$.

Next, we consider composition.

$$\begin{aligned}
\Psi_1 \circ \Psi_2 &= \tilde{\psi}_1 \circ e^{ad(\frac{1}{\nu}H_1)} \circ \tilde{\psi}_2 \circ e^{ad(\frac{1}{\nu}H_2)} \\
&= \widetilde{\psi_1 \circ \psi_2} \circ \left\{ (\widetilde{\psi_1 \circ \psi_2})^{-1} \circ \tilde{\psi}_1 \circ \tilde{\psi}_2 \right\} \\
&\qquad \circ \left\{ \tilde{\psi}_2^{-1} \circ e^{ad\frac{1}{\nu}H_1} \circ \tilde{\psi}_2 \right\} \circ e^{ad(\frac{1}{\nu}H_2)} \\
&= \widetilde{\psi_1 \circ \psi_2} \circ e^{ad\frac{1}{\nu}H_3} \circ e^{ad\frac{1}{\nu}H_1'} \circ e^{ad(\frac{1}{\nu}H_2)} \\
&= \widetilde{\psi_1 \circ \psi_2} \circ e^{ad(\frac{1}{\nu}H(\psi_1, H_1, \psi_2, H_2))},
\end{aligned}$$

where we used

(i) $Diff(M,\omega) \times \underline{Aut}(C_M) \to Aut(C_M)$, smooth,

$$(\psi, \Psi) \longmapsto \tilde{\psi}^{-1} \circ \Psi \circ \tilde{\psi},$$

(ii) $Diff(M,\omega) \times Diff(M,\omega) \to \underline{Aut}(C_M)$, smooth.

$$(\psi, \phi) \longmapsto (\widetilde{\psi \circ \phi})^{-1} \circ \tilde{\psi} \circ \tilde{\phi}.$$

Obviously, $\underline{Aut}(C_M)$ is a Lie subgroup of $Aut(C_M)$.

Using the following sequence

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \underline{Aut}(C_M) & \rightarrow & Aut(C_M) & \rightarrow & Diff(M,\omega) & \rightarrow & 1, \\
 & & & & \tilde{\phi} & \longleftarrow & \phi & &
\end{array}
$$

we completes the proof. ∎