A note on E. Witten: The Verlinde algebra and the cohomology of the Grassmannian; Section 2

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I shall introduce Section 2 of E.Witten's paper ; The Verlinde algebra and the cohomology of the Grassmannian , p. 5 - p.29 , where the relation between the Verlinde algebra and the gauged WZW model of G/G is explained. Witten's description is given by physisist argument, that is, one introduces an action functional and find the symmetry of actions by the infinitesimal change of field variables. The description follows some line of conjectures and is insupportably puzzled for those who are acquainted with the mathematical description. Here I shall try to explain things in mathematics setting whenever I can translate the content of each part of this paper to such style. I must say that I could merely do this job almost half. But I hope you will recognize the space of non-abelian theta functions and the definition of Verlinde algebra; p.5-p.21of [W].

I prepared a small dictionary to see the correspondence between the infinitesimal actions of field variables and the covariant derivations by the action of Lie group. I shall add some higher dimensional generalization which I had no time to mention at the occasion of my talk. Moreover I put an addendum to study classical results on theta devisors.

Many parts of Section 2 of this paper are duplicated from the same author's paper: On holomorphic factorization of WZW and coset models, CMP 144,189-212(1992).

1 Small dictionary of physics notations (infinitesimal descriptions) - our notations (mathematics, global notations).

Witten page 7 ;

$$\text{infinitesimal gauge transformation} \quad = \quad Lie\,\mathcal{G}\text{-action on } \mathcal{A},$$

$$\delta g = -gu \quad \overset{u=\xi}{\Longrightarrow} \quad g = \exp \xi,$$

$$\delta A_i = -D_i\, u \;=-\partial_i u - [A_i, u] \quad \Longrightarrow \quad d_A\xi = d\xi + [A\,,\,\xi\,]$$

Witten formula (2.3), p.6:

$$\frac{D}{DA_i} = \frac{\delta}{\delta A_i} + \frac{i}{4\pi}\epsilon^{ij}A_j \quad \Longrightarrow \quad \nabla_A = \tilde{d} + \theta$$

$$\nabla_{\mathbf{a}}\Phi(A) \quad \Longrightarrow \quad (\partial_A\Phi)a + \theta_A(a)\Phi(A)\,.$$

$$\theta_A(a) = \frac{i}{4\pi}\int_\Sigma Tr(Aa)$$

Witten formula (2.8), p.7:

$$(D_i\frac{D}{DA_i} - \frac{ik}{4\pi}F) \quad \Longrightarrow \quad \xi_\Sigma\, + J^\xi\frac{\partial}{\partial\phi} \;=\; d_A\xi\, + (\,F_A\xi)\frac{\partial}{\partial\phi}$$

$$\int_\Sigma Tr\,\alpha\,(D_i\frac{D}{DA_i} - \frac{ik}{4\pi}F) \quad \overset{?}{\Longrightarrow} \quad \int_\Sigma Tr\,(d_A\xi\, a \;-\; F_A\xi\,)$$

2 The space of connections: gauge potentials

1. Σ: 2-dimensional manifold without boundary.
   $G = SU(n), \quad n \geq 2 \qquad , Lie\, G = su(n)$.
   $P = \Sigma \times G$, or a trivial G-principal bundle.

2. $\mathcal{A}(\Sigma)$: Space of (irredusuble) connections on P.
   $\mathcal{A}(\Sigma)$ is an infinite dimensional affine space modeled by $\Omega^1(\Sigma,\, Lie\, G)$. Hence $A + a \in \mathcal{A}$ for $\forall a \in \Omega^1(\Sigma,\, Lie\, G)$. The tangent space $T_A\mathcal{A}$ at $A \in \mathcal{A}$ is $\Omega^1(\Sigma,\, Lie\, G)$.

3. $$F_A; \qquad \text{curvature} \; ;= dA + \frac{1}{2}[A, A] = dA + A^2, \quad A \in \mathcal{A}$$

4. Differential calculus on the affine space $\mathcal{A}(\Sigma)$ is executed by the Frechet differentiation: For a polynomial $\Phi = \Phi(A)$,
   $$\partial_A \Phi(A) = \lim_{t \longrightarrow 0} \frac{1}{t} \left(\Phi(A + ta) - \Phi(A) \right), \quad \forall a \in T_A\mathcal{A}$$
   For example,
   $$\partial_A A(a) = A\, a, \quad \partial_A F_A(a) = d_A a = da + [A, a], \quad \text{etc.}$$
   Vector fields and differential forms on $\mathcal{A}$ are defined in a usual manner.
   $\tilde{d}$: exterior derivative on the affine space $\mathcal{A}$.
   For a function $(\tilde{d}\Phi)_A a = \partial_A \Phi(a), \quad \forall a \in T_A\mathcal{A}$. For a 1-form
   $$(\tilde{d}\phi)_A(a, b) = \partial_A < \phi, b > (a) - \partial_A < \phi, a > (b) - \phi([a, b]), \quad \forall a, b \in T_A\mathcal{A}$$

5. Group of gauge transformations .
   $$\mathcal{G}(\Sigma) := Aut_0(P) = \quad C^\infty(\Sigma,\, G\,)$$
   Here $Aut_0(P)$ is the group of base point preserving automorphisms on P.
   $\mathcal{G}(\Sigma) \ni g$ acts on $A \in \mathcal{A}(\Sigma)$ by
   $$g \cdot A = g^{-1} A g + g^{-1} dg\,.$$

6. $$\mathcal{M} = \mathcal{A}(\Sigma)/\mathcal{G}(\Sigma), \qquad \text{the moduli space of connections}$$

Various aspects of the moduli space of flat connections

$$\mathcal{A}^\flat = \{A \in \mathcal{A}(\Sigma) : F_A = 0\}, \qquad \text{flat connections}$$

$$\mathcal{M}^\flat = \mathcal{A}^\flat/\mathcal{G}, \text{ the moduli space of flat connections}$$

1.

$$\mathcal{M}^\flat \cong Hom\,(\pi_1(\Sigma), G)\,/G\,.$$

bijective correspondense.

$Hom\,(\pi_1(\Sigma), G)\,/G$ is the moduli space of representations of $\pi_1(\Sigma)$ in the group G
modulus the conjugate representations :

$$g \sim g' \Longleftrightarrow \exists h \in G\ : g = hgh^{-1}$$

2.

$$Hom\,(\pi_1(\Sigma), G)\,/G \cong \text{set of equivalence class of flat } G\text{-bundle.}$$

Here "flat" means (1) a flat vector bundle with constant transition functions, i.e. a local system, and (2) a vector bundle with a flat connection.

3.

$$Hom\,(\pi_1(\Sigma), G)\,/G \quad \cong \quad \mathcal{M}_J^G \quad .$$

$\mathcal{M}_J^G$ is the moduli space of semi-stable holomorphic vector bundle $E \longrightarrow \Sigma$ of rank n with $\det E = \mathcal{O}_\Sigma$.

Here we need a complex structure J on Σ and $\mathcal{O}_\Sigma$ is the structure sheaf on $(\,\Sigma\,,\ J\,)$.

3 symplectic structure and the moment map

1. **symplectic structure**

   ω: 2-form on $\mathcal{A}(\Sigma)$ defined by

   $$\omega_A(a,b) = \frac{i}{2\pi}\int_\Sigma Tr(a\wedge b), \qquad a,b\in T_A\mathcal{A}.$$

   non-degenerate closed 2-form on $\mathcal{A}$.

   - $(\mathcal{A},\omega)$: symplectic space.

2. **Moment mapping for the $\mathcal{G}$- action**

   The infinitesimal action of a $\xi \in Lie\,\mathcal{G}$ on $\mathcal{A}$ is given by the corresponding fundamental vector field:

   $$\xi_{\mathcal{A}}(A) = \frac{d}{dt}|_{t=0}(\exp\, t\xi)\cdot A \ = d_A\xi = d\xi + [A,\xi] \quad \in T_A\mathcal{A}.$$

   - The moment mapping of this action is given by

   $$\mathcal{A}\ni A \xrightarrow{J} J(A) = F_A \in (Lie\mathcal{G})^*.$$

   In fact,

   $$\begin{aligned} J^\xi(A) &= \frac{i}{2\pi}\int_\Sigma Tr(F_A\,\xi), \quad \text{for } \xi\in Lie\,\mathcal{G} \\ (\widetilde{d}\,J^\xi)_A a &= \frac{i}{2\pi}\int_\Sigma Tr(d_A a\,\xi) = \frac{i}{2\pi}\int_\Sigma Tr(a\,d_A\xi) = \omega_A(a,d_A\xi). \end{aligned}$$

   $\square$

   $$J^{-1}(0) = \{A:\ F_A = 0\} = \mathcal{A}^\flat; \qquad \text{flat connections}$$

4 Chern-Simons functional

1. N: a 3-manifold with $\partial N = \Sigma$.
   For $\tilde{A}\in\mathcal{A}(N)$, Chern-Simons functional is defined by

   $$CS(\tilde{A}) = \frac{1}{4\pi}\int_N Tr(\tilde{A}\tilde{F} - \frac{1}{3}\tilde{A}^3), \qquad .$$

2. By the action of a gauge transformation CS changes as

$$
\begin{aligned}
(\delta\, CS)(\tilde{g}, \tilde{A}) &= CS(\tilde{g} \cdot \tilde{A}) - CS(\tilde{A}) \\
&= \frac{1}{4\pi} \int_{\Sigma} Tr\,(\,dg \cdot g^{-1} A) + \Gamma(g),
\end{aligned}
\tag{4.1}
$$

where

$$
\Gamma(g) = \frac{1}{12\pi} \int_{B} Tr(d\tilde{g} \cdot \tilde{g}^{-1})^3 \,,
$$

for $\tilde{g} \in \mathcal{G}(N)$ such that $\tilde{g}|\Sigma = g$.

Since ($\pi_2(G) = 1$) $g \in \mathcal{G}(\Sigma)$ is extended to $\tilde{g} \in \mathcal{G}(N)$, and $\Gamma(g)$ is independent of the extension $\tilde{g}$ by virtue of the fact $H^3(G, \mathbf{Z}) = \mathbf{Z}$.

The RHS of (4.1) depends only on $A = \tilde{A}|\Sigma$ and on $g = \tilde{g}|\Sigma$ by mod $2\pi\mathbf{Z}$.

proof of (4.1)

$$
\begin{aligned}
& CS(\tilde{g} \cdot A) - CS(\tilde{A}) = \\
& \frac{1}{4\pi} \int_N Tr\,[(\tilde{g}^{-1}\tilde{A}\tilde{g} + \tilde{g}^{-1}d\tilde{g})(\tilde{g}^{-1}\tilde{F}\tilde{g}) - \frac{1}{3}(\tilde{g}^{-1}\tilde{A}\tilde{g} + \tilde{g}^{-1}d\tilde{g})^3 \\
& -\tilde{A}\tilde{F} + \frac{1}{3}\tilde{A}^3] \\
& = \frac{1}{4\pi} \int_N Tr\,[d\tilde{g}\tilde{g}^{-1}\tilde{F} - d\tilde{g}\tilde{g}^{-1}\tilde{A}^2 - (d\tilde{g}\tilde{g}^{-1})^2\tilde{A} + \frac{1}{3}(d\tilde{g}\tilde{g}^{-1})^3] \\
& \overset{Stokes}{=} \frac{1}{4\pi} \int_{\Sigma} Tr\,(dg\, g^{-1}A) + \frac{1}{12\pi} \int_N Tr(d\tilde{g} \cdot \tilde{g}^{-1})^3
\end{aligned}
$$

$\square$

3. **Polyakov-Wiegmann**

Put

$$
\begin{aligned}
W(g, A) &= (\delta\, CS)(\tilde{g}, \tilde{A}). \\
&\overset{(4.1)}{=} \frac{1}{4\pi} \int_{\Sigma} Tr\,(\,dg \cdot g^{-1} A) + \Gamma(g).
\end{aligned}
$$

Then $(\delta W)(f, g; A) = 0$, that is,

$$
W(gh, A) = \; W(g,\, h \cdot A) \;+\; W(h, A)\,, \quad \text{mod } 2\pi\mathbf{Z}. \tag{4.2}
$$

5 Pre-quantum line bundle

1.

From Polyakov-Wiegmann (4.2),

exp $i\,W(g,A)$ gives a $U(1)$-cocycle:

$$\exp\, i\,W(f,A)\, \exp\, i\,W(g,\, f\cdot A) = \exp\, i\,W(fg,A).$$

(cocycle condition)

2.

If we define the action of $\mathcal{G}$ on $\mathcal{A}\times \mathrm{C}$ by

$$g\cdot(A,c) = (\, g\cdot A,\, \exp\, i\,W(g,A)c\,)\,,$$

then we have a complex line bundle

$$\mathcal{L} = \mathcal{A}\times \mathrm{C}/\mathcal{G} \xrightarrow{\pi} \mathcal{M} = \mathcal{A}/\mathcal{G},$$

with the transition function

$$\exp\, i\,W(g,A)$$

which is endowed with a hermitian structure (the transition function is in $U(1)$).

3.

Restrict the line bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ to $\mathcal{M}^\flat$.

$$\mathcal{L}^\flat = \pi^{-1}(\mathcal{M}^\flat) = \mathcal{A}^\flat \times \mathrm{C}/\mathcal{G} \longrightarrow \mathcal{A}^\flat/\mathcal{G} = \mathcal{M}^\flat,$$

Proposition 5.1. *1.*

$\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ *is a hermitian line bundle with connection* θ*, that is given by*

$$\theta_A(a) = \frac{i}{4\pi} \int_\Sigma Tr(Aa). \tag{5.1}$$

The curvature of θ *is* $i\omega$*.*

2.

$\mathcal{L}^\flat$ *is a hermitian line bundle with connection* θ *given by the same formula as above*

This is analogous to the well known explanation of line bundle with connection as you see below.

$$\begin{array}{ccccc} 0 & & & & \\ \uparrow d & & & & \\ \omega & \longrightarrow & 0 & & \\ \uparrow d & & \uparrow d & & \\ \theta_\alpha = \frac{ds_\alpha}{s_\alpha} & \xrightarrow{\delta} & d\log f_{\alpha\beta} = \frac{ds_\beta}{s_\beta} - \frac{ds_\alpha}{s_\alpha} = \theta_\beta - \theta_\alpha & \longrightarrow & 0 \\ & & \uparrow d\log & & \\ & & f_{\alpha\beta} = \frac{s_\beta}{s_\alpha} & & \end{array}$$

s_α: local section,
$f_{\alpha\beta}$: $U(1)$-transition function,
θ: connection,
ω: curvture.

$$\delta\theta = d \log f_{\alpha\beta}$$

The proof of Proposition5.1 follows if we show the commutativity of the following diagram of " Line bundle with connection" over $\mathcal{A}/\mathcal{G}$:

$$\begin{array}{ccc} 0 & & \\ \uparrow \tilde{d} & & \\ \omega & \longrightarrow & 0 \\ \uparrow \tilde{d} & & \uparrow \\ \theta_A & \xrightarrow{\delta} & -i\,\tilde{d}\,W(g,A) \\ & & \uparrow \tilde{d}\log \\ & & \exp\, i\,W(g,A) \end{array}$$

Here $\exp ik\,W(g,A)$ is the $U(1)$-transition function, and θ_A is the connection with the curvature ω. In fact

$$\tilde{d}\theta(a,b) = \frac{i}{4\pi}\int_{\Sigma} Tr(ab) - \frac{i}{4\pi}\int_{\Sigma} Tr(ba) = \omega(a,b)$$

We have also

$$\begin{aligned}
(\delta\theta_A(g))a &= g\cdot\theta_A(a) - \theta_A(a) \\
&= \frac{i}{4\pi}\int_{\Sigma} Tr[(g^{-1}Ag - g^{-1}dg)g^{-1}ag - Aa] \\
&= -\frac{i}{4\pi}\int_{\Sigma} Tr[g^{-1}dgg^{-1}ag] = -i\,\tilde{d}\,W(g,A).
\end{aligned}$$

Thus we have the commutativity of the diagram. □

6 Equivariant pre-quantum line bundles

Definition 6.1. (M,ω,Φ): a Hamiltonian G-manifold
$\Longleftrightarrow$ M : equipped with a G-invariant symplectic form ω
• G acts on M with a moment map Φ.

Φ: moment map $\Longleftrightarrow$

$$\begin{aligned}
(i)\quad \Phi \quad \text{equivariant}: &\quad (Ad_g)^*(\Phi\,(m)) = \Phi(g\cdot m), \\
i.e. &\quad \Phi^{Ad_g\xi}\,(m) = \Phi^{\xi}(g\cdot m), \quad \forall\xi\in\mathfrak{g}
\end{aligned}$$

$$\begin{array}{ccc}
M & \xrightarrow{\Phi} & \mathfrak{g}^* \\
g\downarrow & & \downarrow Ad_g^* \\
M & \xrightarrow{\Phi} & \mathfrak{g}^*.
\end{array}$$

$$(ii) \qquad d\,\Phi^{\xi} = i_{\xi_M}\,\omega\,, \quad \forall\xi\in\mathfrak{g}$$

Definition 6.2.
a Hamiltonian G-manifold (M, ω, Φ) is pre-quantizable
$\Longleftrightarrow$
$\exists \quad P \xrightarrow{\pi} M$: G-equivariant $U(1)$-principal bundle
$\exists \quad \Theta$; a G-invariant connection form on P :
such that

$$\begin{aligned} \pi^*\omega &= -\,d\Theta \qquad \text{curvature} \\ \pi^*\,\Phi^\xi &= \Theta(\,\xi_P\,) \qquad \forall \xi \in \mathfrak{g}\,, \end{aligned}$$

where $\quad \xi_P(u) = \frac{d}{dt}|_{t=0}\,(\,\exp\, t\xi\,\cdot u\,),$ $\quad$ lift of $\mathfrak{g}$-action to P.

We know that

$$\xi_P = \xi_P\,|_{\text{horizontal}} + \Phi^\xi\,\frac{\partial}{\partial \phi}\,, \qquad \text{[Kostant]}$$

i.e. the vertical component of ξ_P is Φ^ξ. The horizontal component is defined by $\pi_*\xi_P = \xi_M$.

7 Pre-quantization line bundle on $\mathcal{A}(\Sigma)$

Proposition 7.1. $(\mathcal{A}, \omega, J)$ *: Hamiltonian $\mathcal{G}$-manifold ,*

$$\begin{aligned} \omega(a,b) &= \frac{i}{2\pi}\int_\Sigma Tr(a\wedge b), \quad \forall a,b \in T_A\mathcal{A}. \\ J^\xi(A) &= \frac{i}{2\pi}\int_\Sigma Tr(F_A\xi), \quad \forall \xi \in Lie\,\mathcal{G}. \end{aligned}$$

Proof
In fact we have already seen that
- ω is a $\mathcal{G}$-invariant symplectic form .
- $Lie\,\mathcal{G} \ni \xi$ generates the fundamental vector field $\xi_{\mathcal{A}}(A) = d_A\xi$ on $\mathcal{A}$,

and $\quad (\tilde{d}\,J^\xi)_A = i_{d_A\xi}\omega$. It rests to prove the equivariance:

$$\begin{aligned} J^{Ad_g\xi}(A) &= \frac{i}{2\pi}\int_\Sigma Tr(F_A\,g\xi g^{-1}) = \frac{i}{2_\pi}\int_\Sigma Tr(g^{-1}F_A g\,\xi) = \\ &= \frac{i}{2\pi}\int_\Sigma Tr(F_{g\cdot A}\,\xi) = J^\xi(g\cdot A). \end{aligned}$$

$\square$

Proposition 7.2.
The Hamiltonian G-manifold $(\mathcal{A}, \omega, J)$ *is Pre-quantizable .*

Proof
$\mathcal{P}$: $U(1)$-principal bundle associated with $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$:

$$\begin{aligned} \mathcal{P} &= \mathcal{A} \times U(1)/\mathcal{G}, \\ g \cdot (A,\, c\,) &= (\, g \cdot A\,,\, c \exp iW(g, A)\,). \end{aligned}$$

$$W(g, A) = \frac{1}{4\pi} \int_{\Sigma} Tr(dg \cdot g^{-1} A\,) + \Gamma(g).$$

Fundamental vector field $\xi_{\mathcal{P}}$ on $\mathcal{P}$ generated by the infinitesimal action of $\xi \in Lie\, \mathcal{G}$ is

$$\begin{aligned} (\xi_{\mathcal{P}})_{(A,z)} &= d_A \xi \, + \, J^{\xi}(A) \frac{\partial}{\partial \phi}\,. \qquad \text{[Kostant]} \\ \Theta &= \pi^* \theta + d\phi\,, \qquad \theta \text{ given in (5.1)} \\ \Theta_{(A,c)}(\xi_{\mathcal{P}}) &= J^{\xi}(A) \\ &= \quad c\, \frac{i}{4\pi} \int_{\Sigma} Tr(F_A \xi) \end{aligned}$$

So

$$\begin{aligned} \pi^* J^{\xi} &= \Theta(\xi_P). \\ \tilde{d}\, \Theta &= \pi^* \, \tilde{d}\, \theta \, = -\, \pi^* \omega. \end{aligned}$$

$(\mathcal{A}, \omega, J)$ has the Pre-quantization $(\mathcal{L}, \Theta)$.

8 Why $H^0(\mathcal{M}_J^G,\, \mathcal{L}^k)$ are the space of generalized theta functions?

1. Consider the classic case: $G = U(1)$. Let $\mathcal{M}_J^{U(1)}$ be the moduli space of holomorphic line bundles on the Riemann surface Σ_J of degree 0.
2. $\mathcal{M}_J^{U(1)}$ is isomorphic to

   $$Hom(\pi_1(\Sigma), U(1)) \simeq \check{H}^1(\Sigma, U(1)) \simeq H^1(\Sigma_J, \mathcal{O})/H^1(\Sigma, \mathbf{Z})$$

   a complex g-dimensional torus.

3. Abel-Jacobi theorem:
$\mathcal{M}_J^{U(1)}$ is isomorphic to

$$\mathbf{C}^g/\Lambda : \quad \text{Jacobi variety of } \Sigma_J$$

(See the addendum of this note if you have not yet studied the classic divisor theory.)

4. Let $\mathcal{L} \longrightarrow \mathcal{M}_J^{U(1)}$ be the line bundle given by the Θ divisor over the Jacobi variety.
Let
$H^0(\mathcal{M}_J^{U(1)}, \mathcal{L})$ be *the space of theta functions.*
$H^0(\mathcal{M}_J^{U(1)}, \mathcal{L}^k)$ be *the space of theta functions of level k.*
Then

$$\dim H^0(\mathcal{M}_J^{U(1)}, \mathcal{L}^k) = k^g \,.$$

independent of J.

The non-abelian analogy of the above is called generally *Verlinde formula*.
Let $\mathcal{R} = \mathcal{M}_J^{SU(r)}$ be the moduli space of stable holomorphic principal G-bundle over Σ.
Let $\mathcal{L}$ be the pre-quantum line bundle over $\mathcal{R}$.

We call
(i) $H^0(\mathcal{R}, \mathcal{L}^k)$: *the space of (non abelian) theta functions of level k.*
(ii) $H^0(\mathcal{R}, \mathcal{L}^{\otimes k})$: The space of conformal blocks
(iii) Verlinde formula

$$\dim_{\mathbf{C}} H^0(\mathcal{R}, \mathcal{L}^k) = ?$$

9 Gerassimov-Witten's Strategy for Verlinde formula

Find a convenient description of the action of $\mathcal{G}$ on $H^0(\mathcal{M}_J^{SU(r)}, \mathcal{L}^k)$.

1. A complex structure J on Σ gives a complex structure on $\mathcal{A}$. In fact it is enough to define the action of $\overline{\partial}_A$: $(1,0)$ component of d_A : $\Omega^0(\Sigma, \, Lie\, G) \longrightarrow \Omega^1(\Sigma, \, Lie\, G)$.

   The $\mathcal{G}$ action on $\mathcal{A}$ extends to an action of the complexified gauge transformation group $\mathcal{G}^{\mathbf{C}} = C^\infty(\Sigma, G^{\mathbf{C}})$.

$$\overline{\partial}_A \longrightarrow g^{-1}\, \overline{\partial}_A\, g\,.$$

   Then we introduce the space $\mathcal{A}/\mathcal{G}^{\mathbf{C}} = \mathcal{R}$: the moduli space of stable holomorphic principal G-bundle over Σ_J, written by the same notation as $\mathcal{A}/\mathcal{G} = \mathcal{R}$.

2. Sections over an open set U of $\mathcal{R}$ are the same as $\mathcal{G}^{\mathbf{C}}$ invariant sections over $\pi^{-1}U \subset \mathcal{A}$, $\pi : \mathcal{A} \longrightarrow \mathcal{R}$.

   Hence $H^0(\mathcal{R}, \mathcal{L}^k)$: the space of non-abelian theta functions at level k, is $\simeq$ to the $\mathcal{G}$-invariant (or equivalently $\mathcal{G}^{\mathbf{C}}$-invariant) subspace $\mathcal{T}$ of $H^0(\mathcal{A}, \mathcal{L}^k)$

3. The projection operator $\Pi :\ H^0(\mathcal{A}, \mathcal{L}^k) \longrightarrow \mathcal{T}$ is given by

$$s \longrightarrow \Pi s = \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{G}} \mathcal{D}g\, g^* s$$

   where $\mathcal{D}g$: formal Haar measure on $\mathcal{G}$.

   We have

$$\dim H^0(\mathcal{R}, \mathcal{L}^k) = \dim \mathcal{T} = Trace\, \Pi. \tag{9.1}$$

4. Find the next kernel $K(A, B; g)$, $A, B \in \mathcal{A}$ for each $g \in \mathcal{G}$:

   $K(A, B; g)$ is a section of the line bundle

$$p_1^*(\mathcal{L}^k) \otimes p_2^*(\mathcal{L}^{-k}) \longrightarrow \mathcal{A} \times \mathcal{A}\,,$$

   where $p_i : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are the projections, such that

$$g^* s(A) = \int \mathcal{D}B\ K(A, B; g)\, s(B) \qquad s \in\ H^0(\mathcal{A}, \mathcal{L}^k), \tag{9.2}$$

   $\mathcal{D}B$ is the Liouville measure on the symplectic space $(\mathcal{A}, \omega)$.

5.

$$
\begin{aligned}
Trace\,\Pi &= Trace\left[s \longrightarrow \frac{1}{\mathrm{vol}(\mathcal{G})}\int_{\mathcal{G}} \mathcal{D}g\, g^{*}s\right] && (9.3)\\
&= \frac{1}{\mathrm{vol}(\mathcal{G})}\int_{\mathcal{G}} \mathcal{D}g\,\mathcal{D}A\;\, K(A,A;g)\,. && (9.4)
\end{aligned}
$$

6. From (9.1), (9.2) and (9.3)

$$
\dim H^{0}(\mathcal{R},\mathcal{L}^{k}) = \frac{1}{\mathrm{vol}(\mathcal{G})}\int_{\mathcal{G}} \mathcal{D}g\,\mathcal{D}A\;\, K(A,A;g)
$$

7. The problem is to look for the kernel $K(g,A,B)$.

8. Witten write the following formula of $K(g,A,B)$.

$$
\begin{aligned}
K(A,B;g) &= \exp\left(-kI(g,A,B)\right)\\
I(g,A,B) &= \frac{i}{8\pi}\int_{\Sigma} Tr\, g^{-1}dg\wedge *g^{-1}dg \;-\; \Gamma(g)\\
&- \frac{i}{2\pi}\int_{\sigma} d^{2}z\, Tr.(\,A_{\bar z}g^{-1}\partial_{z}g - B_{z}\partial_{\bar z}g g^{-1} + B_{z}gA_{\bar z}g^{-1}\\
&- \frac{1}{2}A_{z}A_{\bar z} - \frac{1}{2}B_{z}B_{\bar z}\,)
\end{aligned}
$$

We shall follow in the next section (unsatisfactory) the explanation by Witten of how to derive this formula. This is the procedure how to relate a gauge field $A\in\mathcal{A}$ to the WZW action:

$$
I(g) = \frac{i}{8\pi}\int_{\Sigma} Tr\, g^{-1}dg\wedge *g^{-1}dg \;-\; \Gamma(g)
$$

10 WZW-model , GWZW-model , G/H model

1. **WZW-model**

   Action functional of WZW model:

   $$I(g) \;= \tfrac{i}{8\pi}\int_\Sigma Tr\, g^{-1}dg \wedge * g^{-1}dg - \quad \frac{1}{12\pi}\int_B Tr\,(d\tilde{g}\,\tilde{g}^{-1})^3 \,.$$

   $$\text{kinetic term} \quad + \qquad \text{topological term:}\, \Gamma(g)$$

   Invariant under a conformal change of metric.

2. The partition function of the WZW-model:

   $$Z = Z(k, G, \Sigma) = \int \mathcal{D}g \, \exp(-ikI(g)\,)$$

3. **GWZW-model** : *Gauging the WZW model* means *generalizing the theory from the case $g \in Map(\Sigma, G)$ to the case where g is a section of a bundle $X \longrightarrow \Sigma$ with the fiber G and the structure group $G_L \times G_R$ or $H \subset G_L \times G_R$.*

   Here $(a, b) \in G_L \times G_R$ acts on G by $g \longrightarrow agb^{-1}$. We note that $I(g)$ is invariant under $G_L \times G_R$.

4. [Problem] For a connection A of $X \longrightarrow \Sigma$, one aims to find a gauge invariant functional $I(g, A)$ such that

   $$I(g, 0) = I(g) = \frac{i}{8\pi}\int_\Sigma Tr\, g^{-1}dg \wedge * g^{-1}dg \;-\; \Gamma(g)\,,$$

   when X is trivial $X = \Sigma \times G$.

5. G/H **model**

   Gauge invariant extension of the 1st term is easy: We merely change d to d_A.

   $$\frac{i}{8\pi}\int_\Sigma Tr\, g^{-1}dg \wedge * g^{-1}dg \Longrightarrow \frac{i}{8\pi}\int_\Sigma Tr\, g^{-1}d_A g \wedge * g^{-1}d_A g$$

   However $\Gamma(g)$ has no gauge invariant extension unless one restrict to an *anomaly-free* subgroup

   $$H \subset G_L \times G_R$$

and consider bundles $X \longrightarrow \Sigma$ with structure group H.

$H \subset G_L \times G_R$: anomaly-free subgroup $\stackrel{def}{\Longleftrightarrow}$

$$Tr_L\, tt' = Tr_R\, tt', \quad \forall t, t' \in \mathfrak{h} = Lie\, H, \tag{10.1}$$

where Tr_L and Tr_R are the traces in $\mathfrak{g}_L$ and $\mathfrak{g}_R$ (the adjoint representations of G_L and G_R viewed as H-modules) respectively .

For anomally free $H \subset G_L \times G_R$,

$$\begin{aligned} I(g, A) &= \Gamma(g) - \frac{i}{4\pi} \sum_i \int_\Sigma A^i Tr\,[T_{i,L} dgg^{-1} + T_{i,R} dgg^{-1}] \\ &\quad - \frac{i}{8\pi} \sum_{i,j} \int_\Sigma A^i \wedge A^j\, Tr[\cdots] \qquad W(2.25) \end{aligned}$$

The quantum field theories with Lagrangians
$L(g, A) = kI(g, A)$, $k \in \mathbf{N}$, are called G/H models.

6. **G/G model**

   The diagonal embedding of G in $G_L \times G_R$ is always anomally free and we have G/G model:

   $$I(g, A) = \cdots \qquad W(2.27)$$

   For a G/G model we have

   $$\begin{aligned} \dim H^0(\mathcal{R}, \mathcal{L}^k) &= \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{G}} \mathcal{D}g\, \mathcal{D}A\ K(A, A; g) \\ &= \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{G}} \mathcal{D}g\, \mathcal{D}A\ \exp(-kI(g, A))\,. \end{aligned} \tag{10.2}$$

11 The kernel $K(A,B;g)$ - Witten page 14$\sim$

The case $H = G_L \times G_R$ is anomalous .

In this case (10.1) is not satisfied.

No gauge invariant G/H Lagrangian and no G/H quantum field theory, i.e. one can not construct a gauge invariant action $I(g,A)$ extending the WZW action.

But one can find a *best possible* $I(g,A)$ such that the violation of gauge invariance is independent of g and of the conformal structure of Σ .

And the Witten's answer is

$$
\begin{aligned}
K(A,B;g) &= \exp\left(-kI(g,A,B)\right. \\
I(g,A,B) &= \frac{i}{8\pi}\int_\Sigma Tr\, g^{-1}dg\wedge *g^{-1}dg \;-\; \Gamma(g) \\
&- \frac{i}{2\pi}\int_\sigma d^2z\, Tr.(\, A_{\bar z}g^{-1}\partial_z g - B_z\partial_{\bar z}g g^{-1} + B_z g A_{\bar z} g^{-1} \\
&- \frac{1}{2}A_zA_{\bar z} - \frac{1}{2}B_zB_{\bar z}\,)
\end{aligned}
$$

It is not gauge invariant but its change under a gauge transformation is independent of g

?? It is related in a useful way to the geometry of prequantum line bundle by an infinitesimal gauge transformation

$$\delta g = vg - gu\,,\; \delta A = -d_A u,\; \delta B = -d_B v$$

we have

$$\delta\, I(g,A,B) = \frac{1}{4\pi}\int_\Sigma Tr\,[udA - vdB]\,.$$

12 Try to derive the formula for $I(g; A, B)$; yet incomplete

I think if we repeat the argument from section 1 to section 7 for the group $H_L \times G_R$ we would have the extension $K(g; A, B)$ of the transition function $K(g, A) = \exp iI(g, A)$.

Let H be a subgroup of G. $H_L \subset G_L$.
The action of $H_L \times G_R$ on the WZW symmetry group G is not faithful. In fact let $Z(G)$ be a center of G diagonally embedded in $G_L \times G_R$. Then $Z(G)$ acts trivially: $(a, a) \cdot g = aga^{-1} = g$.

$F = H_L \times G_R/Z$ with $Z = H \cap Z(G)$ acts faithfully.

- $P \longrightarrow \Sigma$: F-principal bundle:
where the right action of F on P is given by

$$u \cdot (a, b) = a^{-1}ub\,.$$

The left action of F on H_L, G_R, F is

$$(a, b) \cdot g = agb^{-1}.$$

Then the adjoint bundles are defined by

$$Ad\,P \;=\; P \times_F \; F\,.$$

$$Ad_L P = P \times_F H_L\,, \quad Ad_R P = P \times_F G_R$$

The groups of gauge transformations are

$$\mathcal{F} = \Gamma(\Sigma, Ad\,P) = \mathcal{H}_L \otimes \mathcal{G}_R\,,$$

$$\mathcal{H}_L = \Gamma(\Sigma, Ad_L P)\,, \quad \mathcal{G}_R = \Gamma(\Sigma, Ad_R P)\,.$$

If P is trivial , $\mathcal{F} = C^\infty(\Sigma, F)$,

$$\mathcal{H}_L = C^\infty(\Sigma, H_L)\,, \quad \mathcal{G}_R = C^\infty(\Sigma, G)$$

Infinitesimal gauge transformation group:

$$Lie\,\mathcal{F} = Lie\,\mathcal{H}_L \oplus Lie\,\mathcal{G}_R\,.$$

$\mathcal{A}$: the space of $(Lie\,H_L)$-valued connections.
$\mathcal{B}$: the space of $(Lie\,G_R)$-valued connections.
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$.

1. $\mathcal{C}$ is a symplectic manifold:

$$\omega\left((a_1, b_1)\,,\,(a_2, b_2)\right) = \frac{1}{2\pi}\int_\Sigma Tr\, a_1 \wedge a_2 \;-\; \frac{1}{2\pi}\int_\Sigma Tr\, b_1 \wedge b_2\,.$$

2. The infinitesimal action of $\zeta = (\xi, \eta) \in Lie\,\mathcal{F}$ on $\mathcal{C}$ is given by

$$\zeta_{\mathcal{C}}((A, B)) = \xi_{\mathcal{A}}(A) + \eta_{\mathcal{B}}(B) = -d_A\xi + d_B\eta\,.$$

3. The moment map

$$J^{\zeta}((A, B)) \;=\; J^{\xi}_L(A) - J^{\eta}_R(B)$$

$$J^{\xi}_L(A) = \frac{-1}{2\pi}\int_\Sigma Tr\, F_A\,\xi, \quad J^{\eta}_R(B) = \frac{-1}{2\pi}\int_\Sigma Tr\, F_B\,\eta$$

$$\tilde{d}\, J^{\xi}_L(a) = \omega(a, -d_A\xi), \quad \tilde{d}\, J^{\eta}_R(b) = \omega(b, d_B\eta)$$

4. L-Chern-Simons and R-Chern-Simons

   Not yet verified

5. L-Polyakov=Wiegner and R-Polyakov-Wiegner.

   $\Longrightarrow$ 2-cocycles (Transition functions) of

   Line bundles $\mathcal{L}_L \longrightarrow \mathcal{A}_L/\mathcal{H}_L$, $\mathcal{L}_R \longrightarrow \mathcal{B}_R/\mathcal{G}_R$

   Line bundle over $\mathcal{A}_L \times \mathcal{B}_R/\mathcal{H}_L \times \mathcal{G}_R$

   Not yet verified

6. $I(g; A, B) =$???

   Not yet verified

13

We have explained or tried to explain section 2 of Witten's paper till page 16, to give the Verlinde formula by the kernel function $K(g; A, B) = \exp\, I(g; A, B)$. The following parts of section 2;

[W-p.17: Inclusion of marked points] and

[W-p.19: Relation to Verlinde algebra]

are easy to comprehend.

——— Here is an outline:

1. Let ρ be a representation of a compact Lie group G in a Hilbert space $\mathcal{H}$. The projection operator onto the G-invariant subspace of $\mathcal{H}$ is
$$\Pi = \frac{1}{vol(G)} \int_G Dg\, \rho(g), \tag{13.1}$$
where Dg an invariant measure on G. The trace of Π is the multiplicity with which the trivial representationof G appears in $\mathcal{H}$.

2. Let V be a representation of G;
$$G \ni g \longmapsto \rho_V(g) \in Aut(V).$$
Let $\overline{V}$ be the complex conjugate representation of G. Then the multiplicity with which V appears in $\mathcal{H}$ is the same as the multiplicity with which the trivial representation appears in $\mathcal{H} \otimes \overline{V}$. So we define the projection Π_V onto G-invariant subspace of $\mathcal{H} \otimes \overline{V}$:
$$\Pi_V = \frac{1}{vol(G)} \int_G Dg\, \rho(g) \otimes \rho_{\overline{V}}(g).$$

3. The multiplicity with which V appears in $\mathcal{H}$ is
$$mult(V) = Trace\, \Pi_V. \tag{13.2}$$

Apply this to the gauge transformation group $\mathcal{G}$ of the principal bundle $P \longrightarrow \Sigma$, and $\mathcal{H} = H^0(\mathcal{A}, \mathcal{L}^{\otimes k})$.

Take a point $x \in \Sigma$. For any representation $\rho_V : G \longmapsto Aut(V)$ of G we have the representation $\rho_{x,V} = \rho_V \circ ev_x$ of $\mathcal{G}$, where
$$ev_x : \mathcal{G} \ni g \longrightarrow g(x) \in G.$$

Pick points x_i; $i = 1, \cdots, s$ and representations V_i; $i = 1, \cdots, s$, and let $V = \otimes_i V_i$. $\mathcal{G}$ acts on V by
$$\rho_V = \otimes_i\, \rho_{x_i, V_i}\,.$$

The conjugate is $\rho_{\overline{V}} = \otimes_i \rho_{x_i, \overline{V}_i}$.

- Path integral representation of the multiplicity with which V appears in $\mathcal{H} = H^0(\mathcal{A}, \mathcal{L}^{\otimes K})$.

1.
$$Tr\,[\rho(g) \otimes \rho_{\overline{V}}(g)] = Tr\,\rho(g) \otimes Tr\,\rho_{\overline{V}}(g)$$

We know
$$Tr\,\rho(g) = \int \mathcal{D}A\,K(A, A; g)$$
with $K(A, B; g)$ given by the formula (10.2). On the other hand
$$Tr\,\rho_{\overline{V}}(g) = \Pi_i\,Tr\,\rho_{\overline{V}_i} g(x_i).$$

2. From (13.2),
$$mult(V) = Tr\,\Pi_V = \frac{1}{vol(\mathcal{G})} \int \mathcal{D}g\,\mathcal{D}A\,Tr\,(\rho(g) \otimes \rho_{\overline{V}})$$
$$= \frac{1}{vol(\mathcal{G})} \int \mathcal{D}g\,\mathcal{D}A\ \exp(-kI(g, A)) \cdot \Pi_i Tr \rho_{\overline{V}_i} g(x_i) \tag{13.3}$$
This is the correlation function
$$\langle \Pi_i Tr \rho_{\overline{V}_i} g(x_i) \rangle \tag{13.4}$$
in the GWZW model.

- Relation to the Verlinde algebra

1. Let T be the maximal torus of G. For any representation V of G there is a line bundle $\mathcal{S} \longrightarrow G/T$ such that $H^0(G/T, \mathcal{S}) \simeq V$.

Let $x_1, \cdots, x_s$ be marked points on Σ. We consider the enlarged connection space;
$$\widehat{\mathcal{A}} = \mathcal{A} \times \Pi_{i=1}^s\,(G/T)_i\,,$$
$(G/T)_i$ means that the gauge transformation group $\mathcal{G}$ acts on (G/T) by composition of ev_{x_i} with the natural action of G on G/T : $\rho_{x_i, V} = \rho_{G/T} \circ ev_{x_i} : \quad \mathcal{G} \longmapsto Aut(G/T)$.

Given irreducible representations V_i of G, let $\mathcal{S}_i$ be a line bundle over $(G/T)_i$ such that $H^0((G/T)_i, \mathcal{S}_i) \simeq \overline{V}_i$. Let $V = \otimes_i V_i$.

Define a line bundle $\widehat{\mathcal{L}} \longrightarrow \widehat{\mathcal{A}}$ by
$$\widehat{\mathcal{L}} = \mathcal{L}^{\otimes k}\ \otimes\ (\otimes_i \mathcal{S}_i),$$
Then
$$H^0(\widehat{\mathcal{A}}, \widehat{\mathcal{L}}) = H^0(\mathcal{A}, \mathcal{L}^{\otimes k}) \otimes (\otimes_i \overline{V}_i)$$

2. The multiplicity of (13.3) is equal to the dimension of the $\mathcal{G}$-invariant subspace of $H^0(\widehat{\mathcal{A}}, \widehat{\mathcal{L}})$.

$$mult(V) = Tr\,\Pi_V = \dim\left(\left(\,H^0(\widehat{\mathcal{A}}, \widehat{\mathcal{L}})\,\right)^{\mathcal{G}}\right).$$

3. Let $\mathcal{R}$ be the quotient of $\mathcal{A}$ by the complexified gauge transformation group $\mathcal{G}_{\mathbf{C}}$; $\mathcal{R}$ is the moduli space of stable holomorphic bundles over Σ with parabolic structure (i.e. the structure group is reduced to T at x_i's).

   $\mathcal{G}_{\mathbf{C}}$-invariant line bundles descends to a line bundle, written by the same letter $\mathcal{L}$, whose sections over $\mathcal{R}$ are $\mathcal{G}$ invariant sections of $\widehat{\mathcal{L}}$ over $\mathcal{A}$. So we have

$$H^0(\widehat{\mathcal{R}}, \widehat{\mathcal{L}}\,) = H^0(\widehat{\mathcal{A}}, \widehat{\mathcal{L}}\,)^{\mathcal{G}}$$

   The left hand is the space of non-abelian theta functions

4.

$$\begin{aligned} \dim\, H^0(\widehat{\mathcal{R}}, \widehat{\mathcal{L}}\,) &= \langle \Pi_i Tr \rho_{\overline{V}_i} g(x_i) \rangle \\ &= \text{correlation functional of GWZW; (13.4)} \end{aligned}$$

14 The Verlinde Algebra

Given a level k, the loop group LG of the compct Lie group G has a finite number of isomorphism classes of unitary representations; the space W of highest weights is a distinguished list of isomorphism classes V_α, $\alpha \in W$.

Let X be the $\mathbf{Z}$ module freely generated by the V_α.

We have a natural metric over X; $\quad g(V_\alpha, V_\beta) = 1$ if $V_\alpha = \overline{V}_\beta$, and $= 0$ otherwise.

a multiplication law

$$V_\alpha \cdot V_\beta = \sum_\gamma N_{\alpha\beta}^\gamma V_\gamma \,.$$

can be defined by giving a cubic form $N_{\alpha\beta\gamma} = \sum_\delta g_{\gamma\delta} N_{\alpha\beta}^\delta$. $N_{\alpha\beta\gamma}$ is defined as follows:

Take a genus zero surface with three marked points x_i, $i = 1, 2, 3$, labelled by integrable representations V_{α_i}, $\alpha_i \in W$. As we have investigated hitherto the choice of α_i, $i = 1, 2, 3$, and the level k determine a moduli space $\widehat{\mathcal{R}}$ of holomorphic bundles with parabolic structure and a line bundle $\widehat{\mathcal{L}}$ over $\widehat{\mathcal{R}}$. The structure constants of the Verlinde algebra are

$$N_{\alpha_1,\alpha_2,\alpha_3} = \dim\, H^0(\widehat{\mathcal{R}}, \widehat{\mathcal{L}}). \tag{14.1}$$

In other words;

$$N_{\alpha_1,\alpha_2,\alpha_3} = \langle\, \Pi_{i=1}^3\, Tr \rho_{\overline{V}_{\alpha_i}} g(x_i)\,\rangle \text{ correlation functional of GWZW} \tag{14.2}$$

15 Addendum 1: A comprehensive study of Theta Divisor

15.1 divisor

M : a compact complex manifold.

a divisor is a formal linear combination

$$D = \sum a_i V_i\,, \qquad a_i \in \mathbf{Z}, \quad V_i : \text{irreducible hypersurface.}$$

$$V_i \cap U_\alpha = \{g_{i\alpha} = 0\}, \quad g_{i\alpha} \in \mathcal{O}(U_\alpha)\,; \text{defining function of } V_i \text{ in } U_\alpha.$$

Let V be an irreducible hypersurface, $p \in V$ and g a defining function for V near p: i.e. g is holomorphic near p and $V = \{g = 0\}$ near p. For a horomorphic function f, $\operatorname{ord} f_{V,p}$ is the largest integer a such that $f = g^a h$, $h \in \mathcal{O}_p$. For the defining functions $g_{i\alpha}$, $g_{i\beta}$ of V_i over $U_\alpha \cap U_\beta$, we have evidently $\operatorname{ord} g_{i\beta} = \operatorname{ord} g_{i\alpha}$.

Given a divisor $D = \sum a_i V_i$, $a_i \in \mathbf{Z}$, we have a local family of meromorphic functions

$$f_\alpha = \Pi_i\,(g_{i\alpha})^{a_i} \in \mathcal{M}^*(U_\alpha),$$

It follows immediately that $f_\alpha \equiv f_\beta \bmod \mathcal{O}^*(U_\alpha \cap U_\beta)$. Then a divisor is a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$

Conversely given a local family of meromorphic functions $\{f_\alpha\}$ such that $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ we have $\operatorname{ord}_V f_\alpha = \operatorname{ord}_V f_\beta$ and we can associate the divisor $D = \sum_V \operatorname{ord}_V(f_\alpha) \cdot V$. Thus

(i)

a divisor is a global section of $\mathcal{M}^*/\mathcal{O}^*$.

$$Div(M) = H^0(M, \mathcal{M}^*/\mathcal{O}^*)$$

$$D = \sum_{\text{irreduc. hypersurf. } V,\, V \cap U_\alpha \neq \phi} ord_V\,(f_\alpha)\,\cdot V\,;$$

(ii)

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta) \implies g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

$$\implies \quad \text{a line bundle } L = [D] \text{ with transition function } g_{\alpha\beta}.$$

(iii)

$$\{f_\alpha \in \mathcal{M}^*(U_\alpha)\}_\alpha \quad \text{local data; } f_\alpha = g_{\alpha\beta} f_\beta\,,$$

$$\Longrightarrow \text{ a meromorphic section } s_f \text{ of } L = [(s)]$$

In summary we have shown that L is the line bundle associated to a divisor $D; L = [D]$, if and only if $\exists\ s \in \Gamma(M, \mathcal{M}^*(L))$ such that $(s) = D$.

$D = \sum a_i V_i$ is said to be effective; $D \geq 0$ iff $a_i \geq 0$.

Then L is the line bundle of an effective divisor $D; L = [D]$, if and only if $\exists\ s \in \Gamma(M, \mathcal{O}^*(L))$ such that $(s) = D$.

15.2 Abel-Jacobi

Σ: a compact Riemann surface of genus g.

$\delta_1, \delta_2, \cdots, \delta_{2g}$: 1-cycles on Σ; a basis for $H_1(\Sigma, \mathbf{Z})$.

δ_i, δ_{i+g} intersect once positively and not intersect other δ_j for $j \neq i, i+g$.

$\omega_1, \omega_2, \cdots, \omega_g$; a basis of holomorphic 1-forms; generator of $H^0(\Sigma, \Omega^1)$.

Put

$$\Pi_i = \begin{pmatrix} \int_{\delta_i} \omega_1 \\ \cdot \\ \cdot \\ \int_{\delta_i} \omega_g \end{pmatrix} \in \mathbf{C}^g \,, \quad i = 1, 2, \cdots, 2g \,.$$

$\Pi = (\Pi_1, \cdots, \Pi_{2g})$ is the *period matrix.*

$$\Lambda = \mathbf{Z}\Pi_1 + \cdots + \mathbf{Z}\Pi_{2g} \subset \mathbf{C}^g \,,$$

forms a lattice.

$\mathcal{J}(\Sigma) = \mathbf{C}^g/\Lambda$ is called a *Jacobi variety* of Σ.

Theorem 15.1 (Abel)**.**
Let $Div^0(\Sigma)$ be the divisors on Σ of order 0. Let μ be the abelian sum:

$$\begin{array}{rccl} \mu : & Div^0(\Sigma) & \longrightarrow & \mathcal{J}(\Sigma) \\ & D = \sum (p_\lambda - q_\lambda) & \longrightarrow & \left(\sum \int_{q_\lambda}^{p_\lambda} \omega_1, \cdots, \sum \int_{q_\lambda}^{p_\lambda} \omega_g \right) \quad \mod \Lambda \end{array}$$

Then $D = (f)$ for a meromorphic function on Σ if and only if

$$\mu(D) = 0 \qquad \mod \Lambda \tag{15.1}$$

$Pic^0(\Sigma)$: the group of divisors of degree 0 in Σ modulo linear equivalence;

$$D \sim D' \Longleftrightarrow \exists f \in \mathcal{M}^*(\Sigma), \quad D = D' + (f) \,.$$

(15.1) says that μ factors

$$Div^0(\Sigma) \longrightarrow Pic^0(\Sigma) \xrightarrow{\tilde{\mu}} \mathcal{J}(\Sigma)$$

Actually $\tilde{\mu}$ is an isomorphism.

Theorem 15.2 (Jacobi)**.** *Let Σ be a Riemann surface of genus g and $p_0 \in \Sigma$. For any $\lambda \in \mathcal{J}(\Sigma)$ we can find g points $p_1, \cdots, p_g \in \Sigma$ such that*

$$\mu\left(\sum_i (p_i - p_0)\right) = \lambda$$

i.e. for any vector $\lambda \in \mathbf{C}^g$ we can find $p_1, \cdots, p_g \in \Sigma$ and paths α_i from p_0 to p_i such that

$$\sum_i \int_{\alpha_i} \omega_k = \lambda_k, \quad \forall k.$$

15.3 Theta divisor

Let $V \simeq \mathbf{C}^n$ be a vector space and $\Lambda \subset \mathbf{C}^n$ be a $\mathbf{Z}$-lattice. The quotient $M = V/\Lambda$ becomes a compact complex manifold called an *Abelian variety*.

Let $L = V \times \mathbf{C}/\Lambda$ be the line bundle corresponding to a divisor $\Theta \subset M$; $L = [\Theta]$.

There corresponds a global holomorphic section of the bundle $L = [\Theta]$ which is also called a *theta divisor* and an entire holomorphic function θ (called a *theta function*) on $V \simeq \mathbf{C}^n$ satisfying certain equation according to Λ.

In the following we consider the case $M = \mathcal{J}(\Sigma_J) = \mathbf{C}^g/\Lambda$: Jacobi variety of the Riemann surface Σ, and $L = [\Theta]$ is the line bundle given by the theta divisor Θ on $\mathcal{J}(\Sigma_J)$.

Theorem 15.3.

1. *$H^0(\mathcal{J}(\Sigma_J), L) \simeq \mathbf{C}$: the space of classical theta functions.*

2. *$\dim H^0(\mathcal{J}(\Sigma_J), L^k) = k^g$: the space of classical theta functions of level k.*

16 Addendum 2: Geometric pre-quantization of the moduli space over 4-dim, manifolds

1. M: 4-dimensional manifold with boundary ∂M.
   $G = SU(n), \quad n \geq 2 \qquad , Lie\, G = su(n)$.
   $P = M \times G$, or a trivial G-principal bundle.

2. $\mathcal{A}(M)$: Space of (irredusuble) connections on P.
   $\mathcal{A}(M)$ is an infinite dimensional affine space modeled by $\Omega^1(M,\, Lie\, G)$.
   Hence $A + a \in \mathcal{A}$ for $\forall a \in \Omega^1(M,\, Lie\, G)$.

3. $$F_A = dA + \frac{1}{2}[a, A] \quad \text{curvature}$$

4. The group of gauge transformations :

   $$\mathcal{G}(M) \ = Aut_0(P) = \quad C^\infty(M,\, G\,)$$

   The group of gauge transformations that are identity on the boundary:

   $$\mathcal{G}_0(M) \ = \{g \in \mathcal{G}(M); \quad g|_{\partial M} = \mathrm{Id}\}\,.$$

   $\mathcal{G}(M) \ni g$ acts on $A \in \mathcal{A}(\Sigma)$ by

   $$g \cdot A = g^{-1}Ag + g^{-1}dg\,.$$

5. $$\mathcal{M} = \mathcal{A}(M)/\mathcal{G}_0(M), \qquad \text{the moduli space of connections.}$$

6. $$\mathcal{A}^\flat = \{A \in \mathcal{A}(M) : \ F_A = 0\,\}, \qquad \text{flat connections.}$$
   $$\mathcal{M}^\flat = \mathcal{A}^\flat/\mathcal{G}_0, \text{ the moduli space of flat connections.}$$
   $$\mathcal{M}^\flat \ \cong \ Hom\,(\pi_1(M), G)\,/G\,.$$

   bijective correspondense. $Hom\,(\pi_1(M), G)\,/G$ is the moduli space of representations of $\pi_1(M)$ in the group G modulus the conjugate representations .

1. For each $A \in \mathcal{A}$ we define the following skew-symmetric bilinear form on $T_A\mathcal{A}$:

$$\begin{aligned} \omega_A(a,b) &= \omega_A^0(a,b) + \omega_A'(a,b), \\ \omega_A^0(a,b) &= \frac{1}{8\pi^3}\int_M Tr[(a\wedge b - b\wedge a)\wedge F_A], \\ \omega_A'(a,b) &= -\frac{1}{24\pi^3}\int_{\partial M} Tr[(a\wedge b - b\wedge a)\wedge A], \end{aligned}$$

for $a, b \in T_A\mathcal{A}$.

Theorem 16.1. *ω is a $\mathcal{G}_0$-invariant closed 2-form on $\mathcal{A}$.*

2. There is a notion of Hamiltonian G-manifold also for a presymplectic structure (degenerate ω), [Guillemin et al.]: Let X be a manifold with a smooth actionon it of a Lie group G. Let σ be a closed 2-form on X. A moment map $\Phi : X \longmapsto (Lie\, G)^*$ is a map that is equivariant withrespect to the G-action on X and the coadjoint action on $(Lie\, G)^*$ such that $\Phi^\xi =< \phi, \xi >$ satisfies

$$d\Phi^\xi = i_{\xi_X}\sigma, \quad \forall \xi \in Lie\, G$$

Definition 16.1. (X, σ, Φ) is a Hamiltonian G-manifold and we say that the action of G on (X, σ, Φ) a Hamiltonian action.

3.

Theorem 16.2. *The action of $\mathcal{G}_0$ on $(\mathcal{A}, \omega, \Phi)$ is a Hamiltonian action and the corresponding moment map is given by*

$$\Phi : \mathcal{A} \mapsto (Lie\, \mathcal{G}_0)^* = \Omega^4(M, Lie\, G): \quad A \mapsto F_A^2\,,$$

$$\langle \Phi(A), \xi \rangle = \Phi^\xi(A) = \frac{1}{8\pi^3}\int_M Tr[\,F_A^2\xi\,], \quad \forall \xi \in Lie\, \mathcal{G}_0$$

4. I constructed a geometric pre-quantization, that is, there is a hermitian line bundle with connection over the moduli space of flat connections $\mathcal{M}^\flat$:

$$\mathcal{L} \xmapsto{\pi} \mathcal{M}^\flat,$$

and the curvature of the connection form θ is $\pi^*\omega$.

The moduli space by the action of total gauge $\mathcal{G}(M)$; $\mathcal{A}/\mathcal{G}$ is also investigated. The latter has a deep relation with the 4-dimensinal generalization of WZW-model.

T.Kori: Chern-Simons pre-quantizations over four-manifolds, D.G.A 29(2011),p.670-684.

T.Kori; Four-dimensional Wess-Zumino-Witten actions, J.G.Ph.47(2003),235-258.