Multiplicative convexity

Eckhard Meinrenken

Waseda University, June 2017
Plan

1. Additive and multiplicative eigenvalue inequalities
2. Hamiltonian convexity
3. q-Hamiltonian convexity
4. Ingredients of the proof
5. twisted q-Hamiltonian convexity
For self-adjoint matrix $A \in \text{Mat}_\mathbb{C}(n)$, let $a_1 \geq \cdots \geq a_n$ be its ordered set of eigenvalues. Let $a = (a_1, \ldots, a_n)$ be the eigenvalue tuple.

**Theorem (Horn Convexity)**

The set of all $(a, b, c) \in \mathbb{R}^{3n}$ such that there exists self-adjoint matrices $A, B, C$ with ordered eigenvalue tuples $a, b, c$, and satisfying

$$A + B + C = 0,$$

is a convex polyhedral cone.

The eigenvalue inequalities giving the faces of this polyhedron were conjectured by Horn (1962), and proved by Klyachko (1998).
Multiplicative Horn Convexity

For $A \in \text{SU}(n)$, write eigenvalues as $\exp(2\pi \sqrt{-1}a_i)$ where

$$a_1 \geq \cdots \geq a_n \geq a_1 - 1, \quad \sum_{i=1}^{n} a_i = 0$$

Define the eigenvalue tuple $a = (a_1, \ldots, a_n)$.

**Theorem**

The set of all $(a, b, c) \in \mathbb{R}^{3n}$ such that there exists special unitary matrices $A, B, C$ with eigenvalue tuples $a, b, c$, and satisfying

$$ABC = I,$$

is a convex polytope.

(M-Woodward, 1997.) The inequalities giving the faces of this polytope were obtained by Agnihotri-Woodward (1998), Belkale (2001).
The Jeffrey-Weitsman polytope describes conjugacy classes of \( A_1, A_2, A_3 \in SU(2) \) with \( A_1 A_2 A_3 = I \).
• $G$ compact, simply connected Lie group, $\mathfrak{g} = \text{Lie}(G)$,
• $T$ maximal torus, $\mathfrak{t} = \text{Lie}(T)$,
• $\mathfrak{t}_+ \subset \mathfrak{t}$ positive Weyl chamber,
• $\mathcal{A} \subset \mathfrak{t}_+$ Weyl alcove.

$$\ker(\text{ad}_\xi) = \mathfrak{t}$$

$$\ker(e^{\text{ad}_\xi} - 1) = \mathfrak{t}$$
There is a quotient map

\[ p: \mathfrak{g} \to \mathfrak{t}_+ \]

with fibers the adjoint orbits, \( \mathcal{O}_\xi = \text{Ad}_G(\xi) \).

**Theorem**

The set

\[ \{(\xi_1, \ldots, \xi_r) \in \mathfrak{t}_+ \times \cdots \times \mathfrak{t}_+ | \exists \zeta_i \in \mathcal{O}_{\xi_i}: \zeta_1 + \cdots + \zeta_r = 0\} \]

is a **convex polyhedral cone**.

Determination of facets: Berenstein-Sjamaar (2000), ...
For $G$ compact, simple, simply connected, there is a quotient map

$$p : G \to \mathcal{A}$$

with fibers the conjugacy classes, $C_\xi = \text{Ad}_G(\exp \xi)$.

**Theorem**

The set

$$\{ (\xi_1, \ldots, \xi_r) \in \mathcal{A} \times \cdots \times \mathcal{A} \mid \exists g_i \in C_{\xi_i} : g_1 \cdots g_r = e \}$$

is a convex polytope.

(M-Woodward (1997).)

One gets these convexity results as special cases of convexity theorems for (quasi-)Hamiltonian spaces.
Definition

A Hamiltonian $G$-space $(M, \omega, \Phi)$ is given by invariant symplectic 2-form $\omega \in \Omega^2(M)$ and equivariant moment map $\Phi: M \to g^*$, satisfying

$$\iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle.$$  

Examples:

a) Coadjoint orbits $\mathcal{O} \subset g^*$, with $\Phi$ the inclusion.

b) Cotangent bundles $T^*G$, with $\Phi: T^*G \to g^*$ left trivialization.
Hamiltonian convexity

Quotient map $p : g^* \rightarrow t^*_+$ with fibers the coadjoint orbits.

**Theorem (Hamiltonian convexity)**

Let $(M, \omega, \Phi)$ be a compact, connected Hamiltonian $G$-space. Then

1. the fibers of $\Phi$ are connected, and
2. the image $p(\Phi(M)) \subseteq t^*_+$

is a convex polytope.

Hamiltonian convexity

Remarks on convexity theorem

- For $G = T$, it just says $\Phi(M) \subset t^*$ is a convex polytope.
- Generalizes to non-compact Hamiltonian spaces with proper moment map $\Phi: M \to g^* \rightsquigarrow$ convex polyhedron.
- Horn cone is moment polyhedron for

$$ (T^* G \times \cdots \times T^* G) // G. $$
Let \( \cdot \) invariant inner product on \( \mathfrak{g} \), defining the Cartan 3-form.

\[
\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G).
\]

**Definition (Alekseev-Malkin-M)**

A \( q \)-Hamiltonian \( G \)-space \((M, \omega, \Phi)\) with \( G \)-valued moment map is given by invariant \( \omega \in \Omega^2(M) \) and equivariant \( \Phi: M \to G \), satisfying

1. \( \iota(X_M)\omega = -\frac{1}{2} \Phi^*(\theta^L + \theta^R) \cdot X \)
2. \( d\omega = -\Phi^*\eta \)
3. \( \ker(\omega) \cap \ker(T\Phi) = 0 \).

**Examples:**

a) Conjugacy classes \( \mathcal{C} \subset G \), with \( \Phi \) the inclusion.

b) Moduli spaces of flat connections, \( \Phi = \) boundary holonomy

\[
M(\Sigma^r_g) \xrightarrow{\Phi} G^r
\]
For $G$ simply connected, have quotient map $p: G \to \mathcal{A}$ with fibers the conjugacy classes.

**Theorem (q-Hamiltonian convexity)**

Let $(M, \omega, \Phi)$ be a connected q-Hamiltonian $G$-space. Then

1. the fibers of $\Phi$ are connected, and
2. the image $p(\Phi(M)) \subset \mathcal{A}$ is a convex polytope.

This is due to M-Woodward (1997), in terms of Hamiltonian loop group actions.
Remarks:

- Taking $M = M(\Sigma^1_1)$, get that commutator map
  \[ \Phi: G \times G \to G, \quad (a, b) \mapsto aba^{-1}b^{-1} \]
  has connected fibers. (But $\Phi$ is surjective here.)

- Multiplicative Horn polytope arises from
  \[ M = M(\Sigma^r_0) \xrightarrow{\Phi} G^r. \]
Proof of (q-)Hamiltonian convexity

We’ll follow Lerman-M-Tolman-Woodward 1998.

Let \((M, \omega, \Phi)\) be a compact connected Hamiltonian \(G\)-space.

Lemma

\[ \exists \text{ unique face } \sigma \text{ of } t^*_+ \text{ such that } (p \circ \Phi)^{-1}(\sigma) \subset M \text{ is dense.} \]

Definition

\(Y = \Phi^{-1}(\sigma)\) is called the \text{principal cross-section}.

1. \(Y\) is a smooth, connected submanifold of \(M\).
2. \(Y\) is a Hamiltonian \(T\)-space, with \(\Phi_Y = \iota_Y^* \Phi, \omega_Y = \iota_Y^* \omega\)
3. \(\Phi_Y\) proper as a map to \(\sigma \subset t^*\).
4. \(\Phi_Y(Y) = p(\Phi(M))\).

Similar for q-Hamiltonian spaces, with \(\mathcal{A}\) replacing \(t^*_+\).
Proof of (q-)Hamiltonian convexity

One combines the ‘principal cross-section’ with the following version of abelian convexity:

Let \((Y, \omega_Y, \Phi_Y)\) be a connected Hamiltonian \(T\)-space, with moment map

\[ \Phi_Y : Y \rightarrow t^* \]

valued in convex subset \(U \subset t^*\) and proper as a map to \(U\). Then

1. the fibers of \(\Phi_Y\) are connected, and
2. \(\Phi_Y(Y) \subset t^*\) is the intersection of \(U\) with a convex polyhedral set.

Proved using local normal forms and ‘local-global’ arguments.

(Condeveaux-Dazord-Molino, Hilgert-Neeb-Planck, LMTW, Sjamaar,..).

Conclude convexity properties of \(p(\Phi(M)) = \Phi_Y(Y)\).
Twisted conjugation

Given a group automorphism $\kappa \in \text{Aut}(G)$, define the twisted conjugation action

$$\text{Ad}_g^{(\kappa)}(a) = g a \kappa(g^{-1}).$$

Its orbits are the twisted conjugacy classes.

Example

If $G = \text{SU}(n)$, then $\kappa(g) = g^\top$ is an automorphism, so we’re considering the action

$$A \mapsto g A g^\top.$$

Example

If $U$ is a disconnected compact Lie group, with identity component $G = U_0$, then the $U$-conjugacy classes are disjoint unions of twisted conjugacy classes of $G$. 
Twisted conjugation

Example

Every automorphism of the Dynkin diagram defines an automorphism of $G$, given on Chevalley generators of $\mathfrak{g}^C$ by

$$h_i \mapsto h_i', \quad e_i \mapsto e_i', \quad f_i \mapsto f_i'.$$

Examples of diagram automorphisms:
Remark:

1. If $\kappa' = \text{Ad}_c \circ \kappa$, then the $\kappa$-twisted and $\kappa'$-twisted conjugation actions are related by right translation $r_c$.
2. $\text{Aut}(G)/\text{Inn}(G) \cong \text{diagram automorphisms}$.

Example

If $G = \text{SU}(n)$, then $\kappa(g) = g^\top$ does not come from a Dynkin diagram automorphism, but $\text{Ad}_c \circ \kappa$ with

$$c = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & \ldots & 0 \end{pmatrix}$$

does.
If $\kappa$ is a diagram automorphisms, then it preserves $T$ and $t_+$. 

There is an alcove $\mathcal{A}(\kappa) \subset t^\kappa$ and a quotient map 

$$p^{(\kappa)} : G \rightarrow \mathcal{A}(\kappa)$$

with fibers the twisted conjugacy classes 

$$C^{(\kappa)}_\xi = \text{Ad}^{(\kappa)}_G(\exp \xi).$$

Note: $\mathcal{A}(\kappa)$ is smaller than the alcove for $(G^\kappa, T^\kappa)$. 
Let $\kappa_1, \ldots, \kappa_r$ diagram automorphisms with $\kappa_r \cdots \kappa_1 = 1$. Have quotient maps

$$p^{(\kappa_i)} : G \to A^{(\kappa_i)}$$

with fibers the $\kappa_i$-twisted conjugacy classes.

**Theorem (M., 2016)**

The set

$$\{ (\xi_1, \ldots, \xi_r) \in A^{(\kappa_1)} \times \cdots A^{(\kappa_r)} | \exists g_i \in C^{(\kappa_i)}_{\xi_i} : g_1 \cdots g_r = e \}$$

is a convex polytope.

Defining inequalities ??? (I don’t know, in general.)
Let $\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L]$. Write $G_\kappa := G$, as a $G$-manifold under twisted conjugation.

**Definition**

A quasi-Hamiltonian $G$-space $(M, \omega, \Phi)$ with $G_\kappa$-valued moment map is given by invariant $\omega \in \Omega^2(M)$ and equivariant $\Phi: M \to G_\kappa$, satisfying

1. $\iota(X_M)\omega = -\frac{1}{2} \Phi^*(\theta^R \cdot X + \theta^L \cdot \kappa(X))$
2. $d\omega = -\Phi^*\eta$
3. $\ker(\omega) \cap \ker(T\Phi) = 0$. 

Eckhard Meinrenken

Multiplicative convexity
Basic examples of spaces with $G\kappa$-valued moment maps:

- Twisted conjugacy classes
- Twisted moduli spaces on surfaces with boundary
- Products (‘fusion’)

If $\kappa' = \text{Ad}_c \circ \kappa$, get $G\kappa'$-valued moment map

$$\Phi' = \Phi_c.$$

$\Rightarrow$ Enough to consider diagram automorphisms.
Let $\kappa \in \text{Aut}(G)$ be a diagram automorphism.

**Theorem**

Let $(M, \omega, \Phi)$ be a $q$-Hamiltonian space with $G_\kappa$-valued moment map. Then the set of twisted conjugacy classes appearing in $\Phi(M)$ form a convex polytope

$$\Delta = p^{(\kappa)}(\Phi(M)) \subset t^\kappa.$$

The proof is similar to the untwisted case; uses ‘cross-sections’.

The result about products of twisted conjugacy classes follows by applying this to a suitably twisted moduli space of the $r$-hold sphere.
Thanks.