Multiplicative convexity

Eckhard Meinrenken

Waseda University,
June 2017

Plan

1. Additive and multiplicative eigenvalue inequalities
2. Hamiltonian convexity
3. q-Hamiltonian convexity
4. Ingredients of the proof
5. twisted q-Hamiltonian convexity

Horn Convexity

For self-adjoint matrix $A \in \mathrm{Mat}_{\mathbb{C}}(n)$, let $a_1 \geq \cdots \geq a_n$ be its ordered set of eigenvalues. Let $a = (a_1, \ldots, a_n)$ be the eigenvalue tuple.

Theorem (Horn Convexity)

The set of all $(a, b, c) \in \mathbb{R}^{3n}$ such that there exists self-adjoint matrices A, B, C with ordered eigenvalue tuples a, b, c, and satisfying

$$A + B + C = 0,$$

is a convex polyhedral cone.

The eigenvalue inequalities giving the faces of this polyhedron were conjectured by Horn (1962), and proved by Klyachko (1998).

Multiplicative Horn Convexity

For $A \in \mathrm{SU}(n)$, write eigenvalues as $\exp(2\pi\sqrt{-1}a_i)$ where

$$, \quad a_1 \geq \cdots \geq a_n \geq a_1 - 1, \;\; \sum_{i=1}^{n} a_i = 0$$

Define the eigenvalue tuple $a = (a_1, \ldots, a_n)$.

Theorem

The set of all $(a, b, c) \in \mathbb{R}^{3n}$ such that there exists special unitary matrices A, B, C with eigenvalue tuples a, b, c, and satisfying

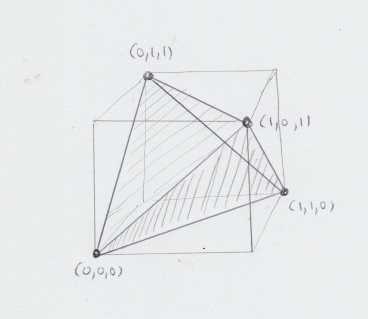
$$ABC = I,$$

is a convex polytope.

(M-Woodward, 1997.) The inequalities giving the faces of this polytope were obtained by Agnihotri-Woodward (1998), Belkale (2001).

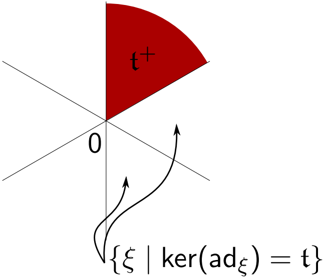
Multiplicative Horn Convexity

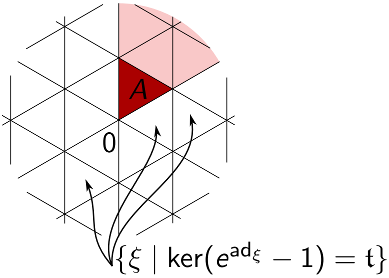
The Jeffrey-Weitsman polytope describes conjugacy classes of $A_1, A_2, A_3 \in \mathrm{SU}(2)$ with $A_1 A_2 A_3 = I$.



General compact groups: Notation

- G compact, simply connected Lie group, $\mathfrak{g} = \text{Lie}(G)$,
- T maximal torus, $\mathfrak{t} = \text{Lie}(T)$,
- $\mathfrak{t}_+ \subset \mathfrak{t}$ positive Weyl chamber,
- $\mathfrak{A} \subset \mathfrak{t}_+$ Weyl alcove.





Sums of adjoint orbits

There is a quotient map

$$p\colon \mathfrak{g} \to \mathfrak{t}_+$$

with fibers the adjoint orbits, $\mathcal{O}_\xi = \mathrm{Ad}_G(\xi)$.

Theorem

The set

$$\{(\xi_1, \ldots, \xi_r) \in \mathfrak{t}_+ \times \cdots \times \mathfrak{t}_+ |\ \exists \zeta_i \in \mathcal{O}_{\xi_i} \colon \zeta_1 + \cdots + \zeta_r = 0\}$$

is a convex polyhedral cone.

Determination of facets: Berenstein-Sjamaar (2000), ...

Products of conjugacy classes

For G compact, simple, simply connected, there is a quotient map

$$p \colon G \to \mathfrak{A}$$

with fibers the conjugacy classes, $\mathcal{C}_\xi = \mathrm{Ad}_G(\exp \xi)$.

Theorem

The set

$$\{(\xi_1, \ldots, \xi_r) \in \mathfrak{A} \times \cdots \times \mathfrak{A} |\ \exists g_i \in \mathcal{C}_{\xi_i} \colon g_1 \cdots g_r = e\}$$

is a convex polytope.

(M-Woodward (1997).)

Determination of facets: Teleman-Woodward (1999).

One gets these convexity results as special cases of convexity theorems for (quasi-)Hamiltonian spaces.

Hamiltonian convexity

Definition

A Hamiltonian G-space (M, ω, Φ) is given by invariant symplectic 2-form $\omega \in \Omega^2(M)$ and equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$, satisfying

$$\iota(\xi_M)\omega = -\mathrm{d}\langle \Phi, \xi \rangle.$$

Examples:

a) Coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, with Φ the inclusion.

b) Cotangent bundles T^*G, with $\Phi\colon T^*G \to \mathfrak{g}^*$ left trivialization.

Hamiltonian convexity

Quotient map $p\colon \mathfrak{g}^* \to \mathfrak{t}^*_+$ with fibers the coadjoint orbits.

Theorem (Hamiltonian convexity)

Let (M, ω, Φ) be a compact, connected Hamiltonian G-space. Then

1. *the fibers of Φ are connected, and*
2. *the image*
$$p(\Phi(M)) \subset \mathfrak{t}^*_+$$
is a convex polytope.

G abelian: Atiyah (1982), Guillemin-Sternberg (1982).

G non-abelian: Guillemin-Sternberg (1983), Kirwan (1984).

Remarks on convexity theorem

- For $G = T$, it just says $\Phi(M) \subset \mathfrak{t}^*$ is a convex polytope.
- Generalizes to non-compact Hamiltonian spaces with proper moment map $\Phi\colon M \to \mathfrak{g}^* \rightsquigarrow$ convex *polyhedron*.
- Horn cone is moment polyhedron for

$$(T^*G \times \cdots \times T^*G)/\!\!/G.$$

quasi-Hamiltonian convexity

Let $\cdot$ invariant inner product on $\mathfrak{g}$, defining the Cartan 3-form.

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G).$$

Definition (Alekseev-Malkin-M)

A q-Hamiltonian G-space (M, ω, Φ) with G-valued moment map is given by invariant $\omega \in \Omega^2(M)$ and equivariant $\Phi\colon M \to G$, satisfying

1. $\iota(X_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot X$
2. $\mathrm{d}\omega = -\Phi^*\eta$
3. $\ker(\omega) \cap \ker(T\Phi) = 0.$

Examples:

a) Conjugacy classes $\mathcal{C} \subset G$, with Φ the inclusion.

b) Moduli spaces of flat connections, $\Phi =$ boundary holonomy

$$M(\Sigma_g^r) \xrightarrow{\Phi} G^r$$

q-Hamiltonian convexity

For G simply connected, have quotient map $p\colon G \to \mathfrak{A}$ with fibers the conjugacy classes.

Theorem (q-Hamiltonian convexity)

Let (M, ω, Φ) be a connected q-Hamiltonian G-space. Then

1. *the fibers of Φ are connected, and*
2. *the image*

$$p(\Phi(M)) \subset \mathfrak{A}$$

is a convex polytope.

This is due to M-Woodward (1997), in terms of Hamiltonian loop group actions.

q-Hamiltonian convexity

Remarks:

- Taking $M = M(\Sigma_1^1)$, get that commutator map

$$\Phi\colon G \times G \to G,\ (a,b) \mapsto aba^{-1}b^{-1}$$

  has connected fibers. (But Φ is surjective here.)
- Multiplicative Horn polytope arises from

$$M = M(\Sigma_0^r) \xrightarrow{\Phi} G^r.$$

Proof of (q-)Hamiltonian convexity

We'll follow Lerman-M-Tolman-Woodward 1998.

Let (M, ω, Φ) be a compact connected Hamiltonian G-space.

Lemma

$\exists$ *unique face* σ *of* $\mathfrak{t}^*_+$ *such that* $(p \circ \Phi)^{-1}(\sigma) \subset M$ *is dense.*

Definition

$Y = \Phi^{-1}(\sigma)$ is called the principal cross-section.

1. Y is a smooth, connected submanifold of M.
2. Y is a Hamiltonian T-space, with $\Phi_Y = \iota_Y^* \Phi$, $\omega_Y = \iota_Y^* \omega$
3. Φ_Y proper as a map to $\sigma \subset \mathfrak{t}^*$.
4. $\overline{\Phi_Y(Y)} = p(\Phi(M))$.

Similar for q-Hamiltonian spaces, with $\mathfrak{A}$ replacing $\mathfrak{t}^*_+$.

Proof of (q-)Hamiltonian convexity

One combines the 'principal cross-section' with the following version of abelian convexity:

Let (Y, ω_Y, Φ_Y) be a connected Hamiltonian T-space, with moment map

$$\Phi_Y \colon Y \to \mathfrak{t}^*$$

valued in *convex* subset $U \subset \mathfrak{t}^*$ and proper as a map to U. Then

1. the fibers of Φ_Y are connected, and
2. $\Phi_Y(Y) \subset \mathfrak{t}^*$ is the intersection of U with a convex polyhedral set.

Proved using local normal forms and 'local-global' arguments. (Condeveaux-Dazord-Molino, Hilgert-Neeb-Planck, LMTW, Sjamaar,..).

Conclude convexity properties of $p(\Phi(M)) = \overline{\Phi_Y(Y)}$.

Twisted conjugation

Given a group automorphism $\kappa \in \mathrm{Aut}(G)$, define the twisted conjugation action

$$\mathrm{Ad}_g^{(\kappa)}(a) = g\, a\, \kappa(g^{-1}).$$

Its orbits are the twisted conjugacy classes.

Example

If $G = \mathrm{SU}(n)$, then $\kappa(g) = g^\top$ is an automorphism, so we're considering the action

$$A \mapsto g\, A\, g^\top.$$

Example

If U is a disconnected compact Lie group, with identity component $G = U_0$, then the U-conjugacy classes are disjoint unions of twisted conjugacy classes of G.

Twisted conjugation

Example

Every automorphism of the Dynkin diagram defines an automorphism of G, given on Chevalley generators of $\mathfrak{g}^{\mathbb{C}}$ by

$$h_i \mapsto h_{i'}, \quad e_i \mapsto e_{i'}, \quad f_i \mapsto f_{i'}.$$

Examples of diagram automorphisms:

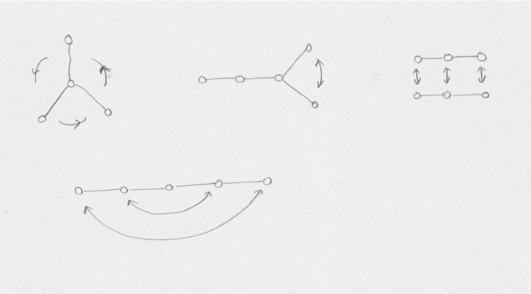


Diagram automorphisms

Remark:

1. If $\kappa' = \mathrm{Ad}_c \circ \kappa$, then the κ-twisted and κ'-twisted conjugation actions are related by right translation r_c.
2. $\mathrm{Aut}(G)/\mathrm{Inn}(G) \cong$ diagram automorphisms.

Example

If $G = \mathrm{SU}(n)$, then $\kappa(g) = g^\top$ does not come from a Dynkin diagram automorphism, but $\mathrm{Ad}_c \circ \kappa$ with

$$c = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \ldots & \ldots & 0 \end{pmatrix}$$

does.

Twisted conjugation

If κ is a diagram automorphisms, then it preserves T and $\mathfrak{t}_+$.

There is an alcove $\mathfrak{A}^{(\kappa)} \subset \mathfrak{t}^\kappa$ and a quotient map

$$p^{(\kappa)}\colon\, G \to \mathfrak{A}^{(\kappa)}$$

with fibers the twisted conjugacy classes

$$\mathcal{C}_\xi^{(\kappa)} = \mathrm{Ad}_G^{(\kappa)}(\exp \xi).$$

Note: $\mathfrak{A}^{(\kappa)}$ is smaller than the alcove for (G^κ, T^κ).

Products of twisted conjugacy classes

Let $\kappa_1, \ldots, \kappa_r$ diagram automorphisms with $\kappa_r \cdots \kappa_1 = 1$. Have quotient maps

$$p^{(\kappa_i)} \colon G \to \mathfrak{A}^{(\kappa_i)}$$

with fibers the κ_i-twisted conjugacy classes.

Theorem (M., 2016)

The set

$$\left\{(\xi_1, \ldots, \xi_r) \in \mathfrak{A}^{(\kappa_1)} \times \cdots \mathfrak{A}^{(\kappa_r)} \mid \exists g_i \in \mathcal{C}_{\xi_i}^{(\kappa_i)} \colon g_1 \cdots g_r = e\right\}$$

is a convex polytope.

Defining inequalities ??? (I don't know, in general.)

Twisted quasi-Hamiltonian spaces

Let $\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L]$. Write $G\kappa := G$, as a G-manifold under twisted conjugation.

Definition

A quasi-Hamiltonian G-space (M, ω, Φ) with $G\kappa$-valued moment map is given by invariant $\omega \in \Omega^2(M)$ and equivariant $\Phi \colon M \to G\kappa$, satisfying

1. $\iota(X_M)\omega = -\frac{1}{2}\Phi^*(\theta^R \cdot X + \theta^L \cdot \kappa(X))$
2. $\mathrm{d}\omega = -\Phi^*\eta$
3. $\ker(\omega) \cap \ker(T\Phi) = 0.$

Twisted quasi Hamiltonian spaces

Basic examples of spaces with $G\kappa$-valued moment maps:

- Twisted conjugacy classes
- Twisted moduli spaces on surfaces with boundary
- Products ('fusion')

If $\kappa' = \mathrm{Ad}_c \circ \kappa$, get $G\kappa'$-valued moment map

$$\Phi' = \Phi c.$$

$\Rightarrow$ Enough to consider diagram automorphisms.

Twisted quasi-Hamiltonian spaces

Let $\kappa \in \mathrm{Aut}(G)$ be a diagram automorphism.

Theorem

Let (M, ω, Φ) be a q-Hamiltonian space with $G\kappa$-valued moment map. Then the set of twisted conjugacy classes appearing in $\Phi(M)$ form a convex polytope

$$\Delta = p^{(\kappa)}(\Phi(M)) \subset \mathfrak{t}^{\kappa}.$$

The proof is similar to the untwisted case; uses 'cross-sections'.

The result about products of twisted conjugacy classes follows by applying this to a suitably twisted moduli space of the r-hold sphere.

Thanks.