Introduction to G-valued moment maps

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Reminder: $\mathfrak{g}^*$-valued moment maps

Hamiltonian G-spaces

A *Hamiltonian G-space* is symplectic manifold (M, ω) with equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$ satisfying

$$\iota(X_M)\omega = -\mathrm{d}\langle \Phi, X \rangle.$$

Examples:

1. Coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, with $\Phi\colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ and
$$\omega(X_{\mathcal{O}}, Y_{\mathcal{O}})|_\mu = \langle \mu, [X, Y] \rangle.$$
2. $G \circlearrowright Q \rightsquigarrow G \circlearrowright M = T^*Q$.
3. $G \subseteq \mathrm{U}(n+1)$ acting on complex submanifold $M \subseteq \mathbb{C}P(n)$.

G-valued moment maps

$\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ Maurer-Cartan forms

$\cdot$ invariant inner product on $\mathfrak{g}$

$\rightsquigarrow$ Cartan 3-form $\eta \in \Omega^3(G)$, $\mathrm{d}\eta = 0$

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G).$$

Definition (Alekseev-Malkin-M)

A q-Hamiltonian G-space (M, ω, Φ) with G-valued moment map is given by invariant $\omega \in \Omega^2(M)$ and equivariant $\Phi \colon M \to G$, satisfying

1. $\iota(X_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot X$
2. $\mathrm{d}\omega = -\Phi^*\eta$
3. $\ker(\omega) \cap \ker(T\Phi) = 0.$

Examples:

1. Conjugacy classes $\mathcal{C} \subset G$, with $\Phi\colon \mathcal{C} \hookrightarrow G$ and

$$\omega(X_\mathcal{O}, Y_\mathcal{O})|_g = \frac{1}{2}(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})X \cdot Y.$$

2. Surface group representations: $\Sigma = \Sigma_{\mathrm{g}}^1$ compact oriented surface with boundary,

$$M_G(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), G) \xrightarrow{\Phi} G,$$

with Φ evaluation on boundary loop. (Similar for more bdry components; also 'twisted' versions.)

3. $M_G(\Sigma_1^1)$ is isomorphic to the *double* $D(G) \cong G \times G$,

$$\Phi(a, b) = aba^{-1}b^{-1}.$$

G-valued moment maps

4. If $\mathcal{M}$ is a Hamiltonian LG-space with proper moment map

$$\Psi\colon \mathcal{M} \to L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g}),$$

equivariant for *gauge action* $g \cdot \mu = \mathrm{Ad}_g\, \mu - \partial g g^{-1}$, get Hamiltonian G-space

$$\Phi\colon M = \mathcal{M}/L_0 G \to G = L\mathfrak{g}^*/L_0 G.$$

5. Multiplicity free examples classified by Knop (2016), including
   - $\mathrm{SU}(2) \circlearrowright S^4$, $\mathrm{Gr}_{\mathbb{R}}(2,4)$ (Alekseev-M-Woodward)
   - $\mathrm{SU}(n) \circlearrowright S^{2n}$ (Hurtubise-Jeffrey-Sjamaar)
   - $\mathrm{Sp}(n) \circlearrowright \mathrm{Gr}_{\mathbb{H}}(k, n+1)$ (Eshmatov, Knop, Paulus)

Products

Hamiltonian G-spaces

$$(M_1 \times M_2,\ \omega_1 + \omega_2,\ \Phi_1 + \Phi_2).$$

q-Hamiltonian G-spaces

$$(M_1 \times M_2,\ \omega_1 + \omega_2 + \tfrac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R,\ \Phi_1\Phi_2).$$

Example:

$$M_G(\Sigma_g^1) \cong \underbrace{D(G) \times \cdots \times D(G)}_{g \quad \text{times}}$$

Reduction

Hamiltonian G-spaces

If 0 is regular value of $\Phi\colon M \to \mathfrak{g}^*$ then

$$M/\!\!/G = \Phi^{-1}(0)/G$$

is a symplectic orbifold. (If not, it's a *stratified symplectic space*.) More generally, for $\mathcal{O} \subset \mathfrak{g}^*$ define $M_{\mathcal{O}} = (M \times \mathcal{O}^-)/\!\!/G$.

q-Hamiltonian G-spaces

If e is regular value of $\Phi\colon M \to G$ then

$$M/\!\!/G = \Phi^{-1}(e)/G$$

is a symplectic orbifold. (If not, it's a *stratified symplectic space*.) More generally, for $\mathcal{C} \subset G$ define $M_{\mathcal{C}} = (M \times \mathcal{C}^-)/\!\!/G$.

Example: $M_G(\Sigma_g^r; \mathcal{C}_1, \ldots, \mathcal{C}_r)) = \big(D(G)^r \times \mathcal{C}_1^- \times \cdots \times \mathcal{C}_r^-\big)/\!\!/G$.

Convexity

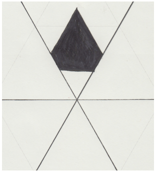
Hamiltonian G-spaces

G compact, $\mathfrak{t}^*_+ \subset \mathfrak{t}^* \subset \mathfrak{g}^*$ Weyl chamber, $p\colon \mathfrak{g}^* \to \mathfrak{t}^*_+$ quotient map.

Theorem (Atiyah, Guillemin-Sternberg, Kirwan): For (M, ω, Φ) a compact, connected Hamiltonian G-space, the fibers of Φ are connected, and

$$p(\Phi(M)) \subset \mathfrak{t}^*$$

is a convex polytope.



Convexity

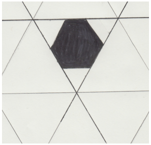
q-Hamiltonian G-spaces

G compact, simply-connected (!), $\mathfrak{A} \subset \mathfrak{t}$ fundamental alcove, $q\colon G \to \mathfrak{A}$ quotient map.

Theorem (M-Woodward): For (M, ω, Φ) a compact, connected q-Hamiltonian G-space, the fibers of Φ are connected, and

$$q(\Phi(M)) \subset \mathfrak{t}$$

is a convex polytope.



Volume forms, DH measures

Hamiltonian G-spaces

Let (M, ω, Φ) a Hamiltonian G-space. We define

- *Liouville volume form:* $\Gamma = \exp(\omega)_{[\dim M]}$
- *Duistermaat-Heckman measure:* For G compact,

$$\mathfrak{m} = \mathcal{R}_{\mathfrak{t}^*}(\Phi_*|\Gamma|) \in \mathcal{D}'(\mathfrak{t}^*)^{W-\mathrm{as}}$$

Here $\mathcal{R}_{\mathfrak{t}^*} : \mathcal{D}'(\mathfrak{g}^*)^G \xrightarrow{\cong} \mathcal{D}'(\mathfrak{t}^*)^{W-\mathrm{as}}$.

- $\mathfrak{m}$ is *piecewise polynomial*
- At regular values, $\mathfrak{m}$ gives volume of reduced space.
- $\langle \mathfrak{m}, e^{2\pi\sqrt{-1}\langle \cdot, \xi \rangle} \rangle$ can be computed by localization (DH formula).

Volume forms, DH measures

q-Hamiltonian G-spaces

Let (M, ω, Φ) a Hamiltonian G-space, G simply-connected. Let

$$\psi = \det{}^{1/2}\left(\frac{1+\mathrm{Ad}_g}{2}\right)\exp\left(\frac{1}{4}\,\frac{1-\mathrm{Ad}_g}{1+\mathrm{Ad}_g}\theta^L\cdot\theta^L\right)\in\Omega(G)^G$$

Theorem (Alekseev-M-Woodward):

$$\Gamma := (e^{\omega}\,\Phi^*\psi)_{[top]}$$

is a *volume form* on M.

In particular, M *even-dimensional* and *orientable*.

Volume forms, DH measures

For G compact, simple, with maximal torus T and one obtains:

Volume of conjugacy classes

The q-Hamiltonian volume of $\mathcal{C} = G.\exp(\nu)$, for $\nu \in \mathrm{int}(\mathfrak{A})$ is

$$\mathrm{vol}(\mathcal{C}) = \frac{1}{\mathrm{vol}(G/T)} \prod_{\alpha>0} 2\sin(\pi\langle\alpha,\nu\rangle)$$

where the product is over positive roots of (G, T).

Compare with:

Volume of coadjoint orbits

The q-Hamiltonian volume of $\mathcal{C} = G.\exp(\nu)$, for $\nu \in \mathrm{int}(\mathfrak{t}^*_+)$ is

$$\mathrm{vol}(\mathcal{O}) = \frac{1}{\mathrm{vol}(G/T)} \prod_{\alpha>0} 2\pi\langle\alpha,\nu\rangle.$$

Volume forms, DH measures

q-Hamiltonian G-spaces

For G compact, simply-connected, define Duistermaat-Heckman measure

$$\mathfrak{m} = \mathcal{R}_T(\Phi_*|\Gamma|)$$

Here $\mathcal{R}_T \colon \mathcal{D}'(G)^G \xrightarrow{\cong} \mathcal{D}'(T)^{W-\mathrm{as}} \cong \mathcal{D}'(\mathfrak{t})^{W^{\mathrm{aff}}-\mathrm{as}}$.

- $\mathfrak{m}$ is *piecewise polynomial*
- At regular values, $\mathfrak{m}$ gives volume of reduced space.
- Localization formula: for all dominant weights $\lambda \in \Lambda^*_+ = \mathfrak{t}^*_+$

$$\langle \mathfrak{m}, t^{\lambda+\rho} \rangle = \dim V_\lambda \sum_F \int_F \frac{e^{\omega_F} \; (\Phi_F)^{\lambda+\rho}}{\mathrm{Eul}(\nu_F, 2\pi i(\lambda+\rho))}$$

(sum over zeroes of $(\lambda+\rho)_M$).

An application to

$$M = G^{2\mathrm{g}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$$

gives the Witten volume formulas for $M(\Sigma_{\mathrm{g}}^r, \mathcal{C}_1, \ldots, \mathcal{C}_r)$.

$$\mathrm{vol}(M(\Sigma_{\mathrm{g}}^r, \underline{\mathcal{C}})) = \frac{\#Z(G)}{\mathrm{vol}(G)^{2-2\mathrm{g}}} \prod_{i=1}^{r} \mathrm{vol}(\mathcal{C}_i) \sum_{\lambda \in \Lambda_+^*} \frac{\prod_{i=1}^r \chi_\lambda(\mathcal{C}_i)}{(\dim V_\lambda)^{2\mathrm{g}+r-2}}$$

Quantization

Hamiltonian G-spaces

Let (M, ω, Φ) a Hamiltonian G-space.

- A *pre-quantization* is Hermitian line bundle $L \to M$ with connection, with Chern form

$$c_1(L) = \omega.$$

  Amounts to integral lift of $[\omega] \in H^2(M, \mathbb{R})$.
- For G simply-connected, get lift of G-action (Kostant).
- $\mathsf{S} = \mathsf{S}^+ \oplus \mathsf{S}^- \to M$ spinor bundle defined by invariant compatible almost complex structure.
- $\rightsquigarrow$ Dirac operator

$$\partial\!\!\!/ \colon \Omega(M, L \otimes \mathsf{S}^{\pm}) \to \Omega(M, L \otimes \mathsf{S}^{\mp})$$

Quantization

Hamiltonian G-spaces

- $\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/) \in R(G)$.
- Multiplicity function: $\mathcal{Q}(M) = \sum_{\mu\in\Lambda^*_+} N(\mu)\,\chi_\mu$.
- Extend N to all of Λ^* by anti-invariance under shifted W-action $w\bullet\mu = w(\mu+\rho)-\rho$. 'Quantum analogue' of DH measure $\mathfrak{m}$.

Theorem ('quantization commutes with reduction'): For $\mu\in\Lambda^*_+$,

$$N(\mu) = \mathcal{Q}(M_{\mathcal{O}}),\quad \mathcal{O} = G.\mu.$$

Quantization

q-Hamiltonian G-spaces

Let (M, ω, Φ) a q-Hamiltonian G-space, G simply-connected, simple.

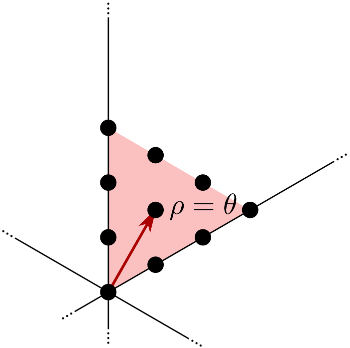
- $d\omega = -\Phi^*\eta$, $d\eta = 0$ means (ω, η) is relative cocycle in $\Omega^3(\Phi)$.
- A pre-quantization is an integral lift of $[(\omega, \eta)]$.
- In particular, η must be integral, i.e. $\cdot$ must be k-th multiple of *basic inner product*. One calls $k \in \mathbb{N}$ the level.

We'll use the basic inner product to identify $\mathfrak{t} \cong \mathfrak{t}^*$.

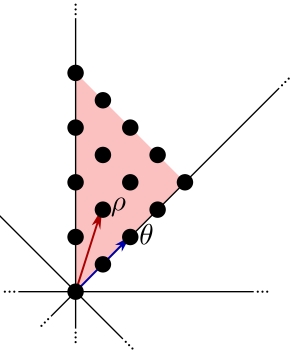
Example: The conjugacy class $\mathcal{C} = G.\exp(\nu)$, $\nu \in \mathfrak{A}$ is level k prequantized if and only if $k\,\nu \in \Lambda^*_+$.

One calls $\Lambda^*_k = \Lambda^* \cap k\mathfrak{A}$ the level k weights.

Pictures of level k weights $\Lambda^*_k = \Lambda^* \cap k\mathfrak{A}$:



$G = \mathrm{SU}(3)$
$k = 3$



$G = \mathrm{Spin}(5)$
$k = 4$

Quantization

Let $R_k(G)$ be the level k fusion ring (Verlinde algebra)

$$R_k(G) = R(G)/I_k(G)$$

with basis $\tau_\mu,\ \mu \in \Lambda_k^*$ the images of $\chi_\mu \in R(G)$.

Let (M, ω, Φ) a q-Hamiltonian G-space.

Fact: A level k prequantization of (M, ω, Φ) gives a canonically defined element

$$\mathcal{Q}(M) \in R_k(G).$$

(As K-homology pushforward (M 2012), or via localization (Alekseev-M-Woodward 1997).)

Quantization

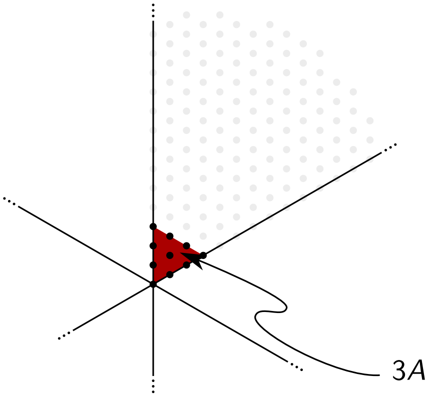
q-Hamiltonian G-spaces

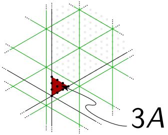
- Multiplicity function: $\mathcal{Q}(M) = \sum_{\mu \in \Lambda_k^*} N(\mu)\, \chi_\mu$.
- Extend N to all of Λ^* by anti-invariance under shifted W^{aff}-action

$$w \bullet \mu = w(\mu + \rho) - \rho.$$

'Quantum analogue' of DH measure $\mathfrak{m} \in \mathcal{D}'(\mathfrak{t})^{W^{\mathrm{aff}}-\mathrm{as}}$.

Quantization





Quantization

Theorem ('quantization commutes with reduction'): Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space, with multiplicity function $N(\mu)$. For all $\mu \in \Lambda_k^*$,

$$N(\mu) = \mathcal{Q}(M_{\mathcal{C}}),$$

where $\mathcal{C} = G.\exp(\mu/k)$.

Proved in AMW 1997 by 'symplectic cutting' method; new proof using norm-square localization due to Loizides (2017).

'$[Q,R]=0$' + 'localization' $\rightsquigarrow$ formula for $\mathcal{Q}(M/\!\!/G)$ as sum of fixed point contributions of

$$t_\lambda = \exp\Big(\frac{\lambda+\rho}{k+h^\vee}\Big), \quad \lambda \in \Lambda_k^*.$$

For $M = G^{2h} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ with

$$\mathcal{C}_i = G.\exp(\mu_i/k), \quad \mu_i \in \Lambda_k^*$$

this is the (symplectic) Verlinde formula:

$$\mathcal{Q}(M(\Sigma_{\mathrm{g}}^r, \underline{\mathcal{C}})) = (\# T_{k+\mathrm{h}^\vee})^{\mathrm{g}-1} \sum_{\lambda\in\Lambda_k^*} |J(t_\lambda)|^{2-2\mathrm{g}} \prod_{i=1}^r \chi_{\mu_i}(t_\lambda)$$

Here J Weyl denominator, and $T_{k+\mathrm{h}^\vee} = ((k+\mathrm{h}^\vee)^{-1}\Lambda^*)/\Lambda$.

Thanks.