Spinor bundles for Hamiltonian loop group spaces

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Outline

Hamiltonian loop group spaces

- G compact Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$
- $LG = \mathrm{Map}(S^1, G)$ loop group (Sobolev class $s > 1/2$)
- $L\mathfrak{g} = \Omega^0_{(s)}(S^1, \mathfrak{g})$
- $\mathcal{A} = \Omega^1_{(s-1)}(S^1, \mathfrak{g})$, with gauge action

$$g.\mu = \mathrm{Ad}_g(\mu) - \mathrm{d}g\ g^{-1}$$

  of the loop group.
- $\cdot$ invariant metric on $\mathfrak{g}$, identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathcal{A} \cong L\mathfrak{g}^*$.

Hamiltonian loop group spaces

Definition

A Hamiltonian loop group space $(\mathcal{M}, \omega, \Psi)$ is an LG-equivariant weakly symplectic Banach manifold, with an LG-equivariant moment map $\Psi \colon \mathcal{M} \to \mathcal{A}$ satisfying

$$\iota(\xi_{\mathcal{M}})\omega = -\mathrm{d}\langle \Psi, \xi \rangle, \qquad \xi \in L\mathfrak{g}.$$

Examples

- Coadjoint orbits $\mathcal{O} \subset \mathcal{A}$
- Moduli spaces of flat connections $\mathcal{M} = \mathcal{M}(\Sigma_{\mathrm{g}}^{1})$
- etc.

Hamiltonian loop group spaces

Every proper Hamiltonian LG-space $(\mathcal{M}, \omega, \Psi)$ determines a quasi-Hamiltonian G-space (M, σ, Φ) with invariant 2-form σ and equivariant moment map

$$\Phi \colon M \to G$$

satisfying certain axioms, including

$$\iota(X_M)\sigma = -\Phi^*\big(\tfrac{1}{2}(\theta^L + \theta^R) \cdot X\big), \qquad X \in \mathfrak{g}.$$

Examples

- $\mathcal{M} = \mathcal{O} \subset \mathcal{A}$ coadjoint orbit $\leftrightarrow$ $M = \mathcal{C} \subset G$ conjugacy class.
- $\mathcal{M} = \mathcal{M}(\Sigma_{\mathrm{g}}^1)$ moduli space $\leftrightarrow$ $M = \mathrm{Hom}(\pi_1(\Sigma_{\mathrm{g}}^1), G) \cong G^{2\mathrm{g}}$.

Hamiltonian loop group spaces

Correspondence:

$$\begin{array}{ccc} & \mathcal{P}G & \\ & {}^{p}\swarrow \quad \searrow^{q} & \\ \mathcal{A} = \mathcal{P}G/G & & G = \mathcal{P}G/LG \end{array}$$

$$\mathcal{P}G = \{\gamma\colon \mathbb{R} \to G |\ \ \gamma(t+1)\gamma(t)^{-1} = \mathrm{const}\}$$
$$p(\gamma) = \gamma^{-1}\partial\gamma, \qquad q(\gamma) = \gamma(1)\gamma(0)^{-1}.$$

Both arrows are principal bundles.

Fact: $\exists\ \varpi \in \Omega^2(\mathcal{P}G)^{LG\times G}$ such that for $\xi \in L\mathfrak{g}$, $X \in \mathfrak{g}$,

$$\iota(\xi_{\mathcal{P}G})\varpi = p^*(\langle \mathrm{d}\mu, \xi\rangle), \qquad \iota(X_{\mathcal{P}G})\varpi = q^*\big(\tfrac{1}{2}(\theta^L + \theta^R)\cdot X\big)$$

while $\mathrm{d}\varpi$ is LG-basic and G-basic.

Hamiltonian loop group spaces

The correspondence

proper Hamiltonian LG-spaces $\leftrightarrow$ quasi-Hamiltonian G-spaces

is described by pullback diagram

$$\begin{array}{ccc} & \mathcal{N} & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ (\mathcal{M}, \omega, \Psi) & & (M, \sigma, \Phi) \end{array}$$

with $q^*\sigma - p^*\omega = \Psi_{\mathcal{N}}^*\varpi$.

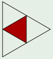
Example (Coadjoint orbits $\leftrightarrow$ Conjugacy classes)

$$\begin{array}{ccc} & \mathcal{N} = (LG \times G).\gamma & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ \mathcal{O} = LG.\mu & & \mathcal{C} = G.a \end{array}$$

Quasi-Hamiltonian spaces

Some other examples of quasi-Hamiltonian spaces:

Example (Multiplicity-free examples)

- $\mathrm{SU}(n) \circlearrowright S^{2n}$ (Jeffrey-Hurtubise-Sjamaar),
- $\mathrm{Sp}(n) \circlearrowright \mathrm{Gr}_{\mathbb{H}}(k, n+1)$ (Eshmatov, Knop, Paulus)
- $\mathrm{SU}(3) \circlearrowright M$ with moment polytope  (Woodward)
- Many other examples (Knop, Paulus)

Example

Wild character varieties (Boalch)

Further examples are obtained using various natural operations. Each corresponds to Hamiltonian LG-space.

Claims

Claim: Every proper Hamiltonian LG-space $(\mathcal{M}, \omega, \Psi)$ has a canonical LG-equivariant Spin_c-structure.

In more detail:

Claims:

1. $\exists$ completion $\overline{T\mathcal{M}}$ on which ω is strongly symplectic.
2. $\exists$ invariant compatible complex structure J on $\overline{T\mathcal{M}}$, in a distinguished 'polarization' class.
3. $\rightsquigarrow$ spinor module $\mathcal{S}_{\overline{T\mathcal{M}}} = \overline{\wedge T^{(0,1)}\mathcal{M}}$ over $\mathbb{C}l(\overline{T\mathcal{M}})$.

Claims

As for the associated quasi-Hamiltonian space:

$$\begin{array}{ccc} & \mathcal{N} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathcal{M} = \mathcal{N}/G & & M = \mathcal{N}/LG \end{array}$$

Claim: $\exists$ 'distinguished' $\widehat{LG} \times G$-equivariant Spin_c-structure on $q^*(TM)$.

Here

$$1 \to \mathrm{U}(1) \to \widehat{LG} \to LG \to 1$$

is the spin central extension of the loop group.

We think of this as a twisted Spin_c-structure on M.

Background: spinor bundles

- $(\mathcal{V}, g)$ Euclidian vector bundle, $\mathrm{rank}(\mathcal{V}) < \infty$ even.
- $\mathbb{C}l(\mathcal{V})$ its $\mathbb{Z}_2$-graded Clifford bundle

Definition

A Spin$_c$-structure on $\mathcal{V}$ is a $\mathbb{Z}_2$-graded spinor module

$$\mathbb{C}l(\mathcal{V}) \circlearrowright \mathcal{S}_{\mathcal{V}}$$

i.e. $\mathbb{C}l(\mathcal{V}) \cong \mathrm{End}(\mathcal{S}_{\mathcal{V}})$. If $\mathcal{V} = TM$ call it Spin$_c$-structure on M.

1. Any two differ by a line bundle $L = \mathrm{Hom}_{\mathbb{C}l(\mathcal{V})}(\mathcal{S}_{\mathcal{V}}, \mathcal{S}'_{\mathcal{V}})$.
2. *Two-out-of-three* for $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$: E.g.,

$$\mathcal{S}_{\mathcal{V}''} = \mathrm{Hom}_{\mathbb{C}l(\mathcal{V}')}(\mathcal{S}_{\mathcal{V}'},\ \mathcal{S}_{\mathcal{V}}).$$

3. Orthogonal complex structure J on $\mathcal{V} \rightsquigarrow \mathcal{S}_{\mathcal{V}} = \wedge(\mathcal{V})^{0,1}$.

Background: spinor bundles

In ∞ dimensions, it's not so easy.

Background: spinor bundles

- $(\mathcal{V}, g)$ separable real Hilbert space
- J orthogonal complex structure $\rightsquigarrow$ spinor module $\mathbb{C}\mathrm{l}(\mathcal{V}) \circlearrowright \mathcal{S}_{\mathcal{V}} = \overline{\wedge \mathcal{V}^{0,1}}$
- $\sim$ equality of bounded operators modulo Hilbert-Schmidt

Theorem (Araki, Shale-Stinespring, others)

The spinor modules for J, J' are isomorphic if and only if $J \sim J'$. In this case, $L = \mathrm{Hom}_{\mathbb{C}\mathrm{l}}(\mathcal{S}_{\mathcal{V}}, \mathcal{S}'_{\mathcal{V}})$ is 1-dimensional.

Consequence: central extension $\widehat{\mathrm{O}_{res}(\mathcal{V})}$ of

$$\mathrm{O}_{res}(\mathcal{V}) = \{g \in \mathrm{O}(\mathcal{V}) |\ g^* J \sim J\}.$$

See also Kac-Peterson ('81), Kostant-Sternberg ('87).

Background: spinor bundles

Application:

Let $\mathbf{L}\mathfrak{g} = L^2$-loops in $\mathfrak{g}$. Fourier decomposition:

$$\mathbf{L}\mathfrak{g}^{\mathbb{C}} = \mathbf{L}\mathfrak{g}^{-} \oplus \mathfrak{g}^{\mathbb{C}} \oplus \mathbf{L}\mathfrak{g}^{+}$$

$\rightsquigarrow$ orthogonal complex structure J on $\mathbf{L}\mathfrak{g} \oplus \mathfrak{g}$

$\rightsquigarrow$ spinor module $\mathcal{S}_{\mathbf{L}\mathfrak{g}\oplus\mathfrak{g}}$

One finds (cf. Pressley-Segal) $LG \subset \mathrm{O}_{res}(\mathbf{L}\mathfrak{g} \oplus \mathfrak{g})$.

$\rightsquigarrow$ central extension $\widehat{LG} \circlearrowright \mathcal{S}_{\mathbf{L}\mathfrak{g}\oplus\mathfrak{g}}$

Let $\widehat{LG}^{\mathrm{spin}}$ = opposite central extension; for G 1-connected, simple get central extension at level $\mathrm{h}^\vee$ = dual Coxeter number.

Background: compatible complex structures

- $\mathcal{V}$ separable real Hilbertable space
- ω strongly symplectic form on $\mathcal{V}$

Definition

A complex structure $J \in \mathbb{B}(\mathcal{V})$ is compatible with ω if $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$ is a Riemannian metric.

Proposition (YLEMYS)

Let J_0, J_1 be ω-compatible complex structures, with metrics g_0, g_1. Suppose $J_0 \sim J_1$. Then

- *J_0, J_1 homotopic through family of ω-compatible complex structures $J_0 \sim J_t$.*
- *$A_t := (-J_t \circ J_0)^{1/2}$ is defined, and takes g_0 to g_t.*

Hamiltonian loop group spaces

Consider first the case of coadjoint orbits $\mathcal{O} \subset \mathcal{A} = L\mathfrak{g}^*$.

$$\omega(\xi_{\mathcal{O}}, \zeta_{\mathcal{O}})|_\mu = \int_{S^1} \partial_\mu \xi \cdot \zeta,$$

$$J = \partial_\mu |\partial_\mu|^{-1}, \quad g(\xi_{\mathcal{O}}, \zeta_{\mathcal{O}})|_\mu = \int_{S^1} |\partial_\mu| \xi \cdot \zeta,$$

on $T_\mu \mathcal{O} \cong L\mathfrak{g}_\mu^\perp \subset L\mathfrak{g}$. Here $\partial_\mu = \partial + \mathrm{ad}_\mu$.

Let $\overline{L\mathfrak{g}}$ be the loops of Sobolev class $\frac{1}{2}$, and $\overline{T_\mu \mathcal{O}} \subset \overline{L\mathfrak{g}}$ completion.

ω, J, g extends to $\overline{T\mathcal{O}}$. The 2-form ω becomes strongly symplectic, and g strongly Riemannian.

Hamiltonian loop group spaces

This works in general:

Theorem (YLEMYS)

Let $(\mathcal{M}, \omega, \Psi)$ be proper Hamiltonian LG-space. Then

1. ω *becomes strongly symplectic on* $\overline{T\mathcal{M}} = (T\mathcal{M} \times \overline{L\mathfrak{g}})/L\mathfrak{g}$.
2. $\exists$ ω*-compatible* LG*-invariant* J *on* $\overline{T\mathcal{M}}$*, within a polarization class (Hilbert-Schmidt equivalence) determined by* $\overline{L\mathfrak{g}} \to \overline{T_x\mathcal{M}}$
3. $\rightsquigarrow$ LG*-equivariant spinor module (essentially unique)*

$$\mathbb{C}l(\overline{T\mathcal{M}}) \circlearrowright \mathcal{S}_{\overline{T\mathcal{M}}}.$$

Q: What about the associated quasi-Hamiltonian space M?

Quasi-Hamiltonian spaces

Recall the correspondence

$$\begin{array}{ccc} & \mathcal{P}G & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathcal{A} = \mathcal{P}G/G & & G = \mathcal{P}G/LG \end{array}$$

Fact: $\exists$ invariant principal connections for both bundles, which extend to invariant splittings of completions:

$$p^* \overline{T\mathcal{A}} \oplus \mathfrak{g} \cong \overline{T(\mathcal{P}G)} \cong q^* TG \oplus \overline{L\mathfrak{g}}$$

Quasi-Hamiltonian spaces

Pull-back $\rightsquigarrow$ invariant principal connections on

$$\begin{array}{ccc} & \mathcal{N} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathcal{M} = \mathcal{N}/G & & M = \mathcal{N}/LG \end{array}$$

Ascent: Connection for p gives

$$(p^*\overline{T\mathcal{M}} \oplus \mathfrak{g}) \oplus \mathfrak{g} \cong \overline{T\mathcal{N}} \oplus \mathfrak{g}.$$

Complex structure on $\overline{T\mathcal{M}}$ $\rightsquigarrow$ complex structure on $\overline{T\mathcal{N}} \oplus \mathfrak{g}$ $\rightsquigarrow$ spinor module

$$\mathbb{C}l(\overline{T\mathcal{N}} \oplus \mathfrak{g}) \circlearrowright \mathcal{S}_{\overline{T\mathcal{N}} \oplus \mathfrak{g}}.$$

Quasi-Hamiltonian spaces

$$\begin{array}{ccc} & \mathcal{N} & \\ & {\scriptstyle p}\swarrow \quad \searrow{\scriptstyle q} & \\ \mathcal{M} = \mathcal{N}/G & & M = \mathcal{N}/LG \end{array}$$

$$\mathbb{C}\mathrm{l}(\overline{T\mathcal{N}} \oplus \mathfrak{g}) \circlearrowright \mathcal{S}_{\overline{T\mathcal{N}}\oplus\mathfrak{g}}.$$

Descent: Connection for q gives

$$\overline{T\mathcal{N}} \oplus \mathfrak{g} = (q^* TM \oplus \overline{L\mathfrak{g}}) \oplus \mathfrak{g}.$$

Use $\mathbb{C}\mathrm{l}(\mathbf{L}\mathfrak{g} \oplus \mathfrak{g}) \circlearrowright \mathcal{S}_{\mathbf{L}\mathfrak{g}\oplus\mathfrak{g}}$ and two-out-of-three to conclude:

Twisted Spin$_c$-structure

$$\mathcal{S}_{q^* TM} := \mathrm{Hom}_{\mathbb{C}\mathrm{l}(\mathbf{L}\mathfrak{g}\oplus\mathfrak{g})}(\mathcal{S}_{\mathbf{L}\mathfrak{g}\oplus\mathfrak{g}}, \mathcal{S}_{\overline{T\mathcal{N}}\oplus\mathfrak{g}})$$

$\widehat{LG}^{\mathrm{Spin}} \times G$-equivariantly. Not so fast....

Quasi-Hamiltonian spaces

To define

$$\mathcal{S}_{q^* TM} := \mathrm{Hom}_{\mathbb{C}\mathrm{l}(\mathbf{L}\mathfrak{g}\oplus\mathfrak{g})}(\mathcal{S}_{\mathbf{L}\mathfrak{g}\oplus\mathfrak{g}}, \mathcal{S}_{\overline{T\mathcal{N}}\oplus\mathfrak{g}}),$$

where

$$\overline{T\mathcal{N}} \oplus \mathfrak{g} = q^* TM \oplus (\overline{L\mathfrak{g}} \oplus \mathfrak{g})$$

we need an *equivariant* isomorphism $\mathcal{N} \times \overline{L\mathfrak{g}} \cong \mathcal{N} \times \mathbf{L}\mathfrak{g}$. We use family

$$\chi(|\partial_\mu|^{1/2}) \colon \overline{L\mathfrak{g}} \to \mathbf{L}\mathfrak{g}$$

for $\mu \in \mathcal{A}$, where $\chi(t) > 0$ for all t, with $\chi(t) = t$ for large t.

Quasi-Hamiltonian spaces

$$\begin{array}{ccc} & \mathcal{N} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathcal{M} = \mathcal{N}/G & & M = \mathcal{N}/LG \end{array}$$

The resulting isomorphism

$$\overline{T\mathcal{N}} \oplus \mathfrak{g} = q^* TM \oplus (\mathbf{L}\mathfrak{g} \oplus \mathfrak{g})$$

'almost' preserves metrics. Now it works:

Twisted Spin$_c$-structure on M

$$\mathcal{S}_{q^* TM} := \mathrm{Hom}_{\mathbb{C}\mathrm{l}(\mathbf{L}\mathfrak{g} \oplus \mathfrak{g})}(\mathcal{S}_{\mathbf{L}\mathfrak{g} \oplus \mathfrak{g}}, \mathcal{S}_{\overline{T\mathcal{N}} \oplus \mathfrak{g}})$$

$\widehat{LG}^{\mathrm{Spin}} \times G$-equivariantly.

Quasi-Hamiltonian spaces

$$\begin{array}{c} \mathcal{N} \\ \big\downarrow q \\ M = \mathcal{N}/LG \end{array}$$

$\widehat{LG}^{\mathrm{Spin}} \times G$-equivariant spinor module

$$\mathbb{C}\mathrm{l}(q^*TM) \circlearrowright \mathcal{S}_{q^*TM}$$

- By construction, the spinor module $\mathbb{C}\mathrm{l}(q^*TM) \circlearrowright \mathcal{S}_{q^*TM}$ is independent of choices, up to homotopy.
- For $\mathcal{M} = \mathcal{O}$ and $M = \mathcal{C}$, can make all choices canonical.
- There are examples of G-conjugacy classes that don't admit Spin_c-structures. For example for $G = \mathrm{Spin}(7)$ or $G = F_4$.
- Local trivializations of $q \rightsquigarrow$ local Spin_c-structures on M, related by 'transition line bundles'.

Abelianization: Hamiltonian G-spaces

Consider first the case of ordinary Hamiltonian G-spaces (M, ω, Φ) with moment map $\Phi\colon M \to \mathfrak{g}^*$.

- Let $T \subset G$ maximal torus.
- Choose positive roots $\rightsquigarrow$ T-invariant complex structure on $\mathfrak{g}/\mathfrak{t}$.

Suppose Φ is transverse to $\mathfrak{t}^*$.

$X = \Phi^{-1}(\mathfrak{t}^*)$ inherits a T-equivariant Spin_c-structure

$$\mathcal{S}_{TX} = \mathrm{Hom}_{\mathbb{C}\mathrm{l}(\mathfrak{g}/\mathfrak{t})}(\mathcal{S}_{\mathfrak{g}/\mathfrak{t}}, \mathcal{S}_{TM}|_X)$$

using $TM|_X = TX \oplus \mathfrak{g}/\mathfrak{t}$.

$$\mathrm{index}_G(S_{TM} \otimes L)|_T = \frac{\mathrm{index}_T(S_{TX} \otimes L|_X)}{\prod_{\alpha>0}(1 - t^{-\alpha})}$$

Abelianization: Hamiltonian LG-spaces

To repeat for loop groups, we need $\mathcal{S}_{\mathbf{L}\mathfrak{g}/\mathfrak{t}}$.

- Positive roots $\rightsquigarrow$ T-invariant complex structure on $\mathbf{L}\mathfrak{g}/\mathfrak{t}$.
- Lattice $\Lambda = \ker(\exp_T)$ is subgroup $\Lambda \subset LG$, preserves $\mathfrak{t} \subset \mathbf{L}\mathfrak{g}$.
- $\rightsquigarrow$ Λ acts on $\mathbf{L}\mathfrak{g}/\mathfrak{t}$, in fact $\Lambda \subset \mathrm{O}_{\mathrm{res}}(\mathbf{L}\mathfrak{g}/\mathfrak{t})$.
- $\rightsquigarrow$ $T \ltimes \widehat{\Lambda}$-equivariant spinor module $\mathcal{S}_{\mathbf{L}\mathfrak{g}/\mathfrak{t}}$.

Abelianization: Hamiltonian LG-spaces

$(\mathcal{M}, \omega, \Psi)$ proper Hamiltonian LG-space, $\mathbb{C}\mathrm{l}(\overline{T\mathcal{M}}) \circlearrowright \mathcal{S}_{\overline{T\mathcal{M}}}$.

Suppose $\Psi\colon \mathcal{M} \to \mathcal{A} = L\mathfrak{g}^*$ is transverse to $\mathfrak{t} \cong \mathfrak{t}^*$.

Then $X = \Psi^{-1}(\mathfrak{t}^*)$ is finite-dimensional pre-symplectic Hamiltonian $T \times \Lambda$-manifold, with $T \ltimes \widehat{\Lambda}$-equivariant Spin$_c$-structure

$$\mathbb{C}\mathrm{l}(TX) \circlearrowright \mathcal{S}_{TX} = \mathrm{Hom}_{\mathbb{C}\mathrm{l}(\mathbf{L}\mathfrak{g}/\mathfrak{t})}(\mathcal{S}_{\mathbf{L}\mathfrak{g}/\mathfrak{t}}, \mathcal{S}_{\overline{T\mathcal{M}}})$$

Given $\mathcal{L} \to \mathcal{M}$, get well-defined index (finite T-multiplicities)

$$\mathrm{index}_T(S_{TX} \otimes \mathcal{L}|_X) \in R^{-\infty}(T);$$

can use this to 'define' $\mathrm{index}_{LG}(\mathcal{S}_{\overline{T\mathcal{M}}} \otimes \mathcal{L})$.

Thanks.