Geometric quantization of the space of connections

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KGIS

1 on Bohr-Somerfeld quantization

1.1 Classical picture

Hamilton's equation on $\mathbf{R}^{2n} \ni (q,p)$, (abbreviated notation for (q,p)!);

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{1.1}$$

defines a flow on the phase space $\mathbf{R}^{2n} \ni (q,p)$: that is, the Hamiltonian vector field

$$X_H = \left(\frac{\partial H}{\partial p} \, , \, -\frac{\partial H}{\partial q} \right)$$

yields the time-evolution of the system

$$(\dot{q}, \dot{p}) = X_H(q,p)$$

Example. Harmonic oscilator.

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2} q^2$$

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -kq \Longrightarrow m\ddot{q} = -kq$$

Two qualitative features of the Hamiltonian description of a system

1. Hamiltonian H is constant along the flow of the Hamiltonian vector field X_H:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$$

2. The divergence of X_H (= the infinitesimal variation of the volume by the vector field X_H) is 0:

$$\nabla \cdot X_H = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}(-\frac{\partial H}{\partial q}) = 0$$

$\Longrightarrow$

Geometry of the Hamiltonian description

1. The level manifold of the hamiltonian is an immersed lagrngian submanifold $L \subset \mathbf{R}^{2n} = T^*\mathbf{R}^n$
2. There is a half-density a on the lagrangian submanifold L.

2 WKB Ansatz

2.1 Approximation of the stationary-phase solution of the Schrödinger equation

Classical Hamiltonian:

$$H(q,p) = \frac{1}{2m}\sum p_i^2 + V(q), \quad (q,p) \in M = \mathbf{R}^{2n}\,.$$

The corresponding Schrödingier operator:

$$\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V\,\cdot\,.$$

Schrödingier operator:

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WKB Ansatz is a method to find an approximation of the stationary-phase solution:

$$\varphi = e^{iS/\hbar}\,a\,,\quad a \sim \sum_{k=0}^{\infty} a_k\,\hbar^k\,,\quad a_0 = 1.$$

of the Schrödinger equation

$$(\widehat{H} - E)\varphi = 0\,.$$

Where $S:\ M \longrightarrow \mathbf{R}$ is the phase function, and a_k is the amplitude.

The $k+1$-st approximate solution is given by

$$e^{iS/\hbar}\sum^{k} a_k\,\hbar^k$$

Hamilton-Jacobi equation

If the phase function $S : M \longrightarrow \mathbf{R}$ satisfies the Hamilton-Jacobi equation

$$H \circ dS = \frac{\|\nabla S\|^2}{2m} + V(q) = E,$$

then the 1-st approximate solution φ_0;

$$(\widehat{H} - E)\varphi = \mathbf{O}(\hbar),$$

is given by

$$\varphi_0 = \exp^{iS/\hbar} \sim 1 + iS/\hbar + \cdots$$

Proof

$$\frac{\partial}{\partial x_j} e^{iS/\hbar} = \frac{i}{\hbar}\frac{\partial S}{\partial x_j} e^{iS/\hbar}, \quad \frac{\partial^2}{\partial x_j^2} e^{iS/\hbar} = \left(\frac{i}{\hbar}\frac{\partial^2 S}{\partial x_j^2} - \frac{1}{\hbar^2}(\frac{\partial S}{\partial x_j})^2\right) e^{iS/\hbar},$$

$$\begin{aligned}(\widehat{H} - E)\varphi_0 &= \left[\frac{\|\nabla S\|^2}{2m} + V(q) - E - \frac{i\hbar}{2m}\Delta S\right] \exp^{iS/\hbar} \\ &= \mathbf{O}(\hbar).\end{aligned}$$

$\square$

2.2 The geometry of the Hamilton-Jacobi equation .

$$L \overset{\text{def.}}{=} image\, dS \; = \; \{(q,p) \in T^*M \; ; \; p_i = \frac{\partial S}{\partial q_i}\} \, .$$

1. From the Hamilton-Jacobi equation: $H \circ dS = E$,
   L is a submanifold of $H^{-1}(E) \subset T^*M$.
   (Note that $H^{-1}(E)$ is $(2n-1)$-dimensional.)

2. Let $\omega = \sum dp_i \wedge dq_i$ be the symplectic form on T^*M. Then, since

   $$\omega|L \; = \; d(\sum p_i dq_i \,) = d(\sum \frac{\partial S}{\partial q_i} \, dq_i \,) = d(dS) = 0,$$

   L is a Lagrangian submanifold, (hence n-dimensional).

3. $\pi : T^*M \;\longrightarrow\; M$ gives the diffeomorphism

   $$\pi_L : \, L \overset{\simeq}{\longrightarrow} \, M.$$

4. The canonical 1-form $\theta = \sum p_i dq_i$ on T^*M induces the 1-form $i^*\theta = d(S \circ \pi_L)$ on L;

   $$\theta|L = \sum \frac{\partial S}{\partial q_i} dq_i \, .$$

Hamilton-Jacobi Theorem

Let H be a function on T^*M . H is locally constant on the Lagrangian submanifold L if and only if the Hamiltonian vector field X_H is tangent to L.

Look at the flow of X_H and the relation

$$dH = \omega(\cdot, X_H)\,, \quad X_H(w) \in T_w L\,, \forall w \in L.$$

2.3 Semi-classical state

For a phase fnction $S = S(x)$ that satisfies the Hamilton-Jacobi equation: $H \circ dS = E$, and for an *amplitude function* $a = a(x)$, we consider the solution of the form:

$$\varphi = \exp^{iS/\hbar} a$$

If a satisfies the *homogeneous transport equation*:

$$a\Delta S + 2\sum \frac{\partial a}{\partial q_j}\frac{\partial S}{\partial q_j} = 0 \tag{2.1}$$

then

$$\varphi_1 = \exp^{iS/\hbar} a$$

gives a 2nd order approximate (stationary phase) solution of the Schr$\ddot{\text{d}}$ingier operator $\widehat{H}$

$$(\widehat{H} - E)\varphi_1 = O(\hbar^2).$$

Proof

$$\frac{\partial}{\partial x_j}(e^{iS/\hbar}a) = \frac{i}{\hbar}\frac{\partial S}{\partial x_j}e^{iS/\hbar}a + e^{iS/\hbar}\frac{\partial a}{\partial x_j}$$

$$\frac{\partial^2}{\partial x_j^2}(e^{iS/\hbar}a) = \left(-\frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x_j}\right)^2 + \frac{i}{\hbar}\frac{\partial^2 S}{\partial x_j^2}\right)e^{iS/\hbar}a + 2\frac{i}{\hbar}\frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j}e^{iS/\hbar} + \frac{\partial^2 a}{\partial x_j^2}e^{iS/\hbar}$$

$$(\widehat{H} - E)\varphi = \left(\frac{1}{2m}\|\nabla S\|^2 + (V - E)\right)\exp^{iS/\hbar} a - \frac{i\hbar}{2m}\left(a\Delta S + 2\sum\frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j}\right)\exp^{iS/\hbar} - \frac{\hbar^2}{2m}\Delta a\,\exp^{iS/\hbar}$$

2.4 Geometry of the semi-classical state

1. For a hamiltonian of the form $H = \sum_i p_i^2/2 + V(q)$, the hamiltonian vector field is

$$X_H = \sum_i (-\frac{\partial V}{\partial q_i})\frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial q_i}.$$

X_H restricted to $L = dS$ becomes

$$X_H|L \;=\; \sum_i (-\frac{\partial V}{\partial q_i})\frac{\partial}{\partial p_i} + \frac{\partial S}{\partial x_i}\frac{\partial}{\partial q_i}.$$

that is,

$$\pi_* X_H = \nabla S. \tag{2.2}$$

2. The homogeneous transport equation

$$a\Delta S + 2\sum \frac{\partial a}{\partial q_j}\frac{\partial S}{\partial q_j} = 0 \qquad \text{implies}$$

$$div\,(a^2\nabla S) \;=\; \sum_j \frac{\partial}{\partial q_j}(a^2\frac{\partial S}{\partial q_j}) = a(a\Delta S + 2\sum \frac{\partial a}{\partial q_j}\frac{\partial S}{\partial q_j}) = 0 \tag{2.3}$$

i.e. the vector field $a^2\nabla S$ is divertence free.

3. By the definition of divergence,

$$(div\,\mathbf{v})|dx| = \mathcal{L}_{\mathbf{v}}|dx|\,, \tag{2.4}$$

where $\mathcal{L}_{\mathbf{v}}$ is the Lie-derivative and $|dx|$ is the volume form.

(**??**): $\pi_* X_H = \nabla S$,
(**??**): $div\,(a^2 \nabla S) = 0$ and
(**??**): $(div\,\mathbf{v})|dx| = \mathcal{L}_{\mathbf{v}}|dx|$
yield

$$\mathcal{L}_{(a^2 X_H)}\,|dx| = \mathcal{L}_{(a^2 \nabla S)}\,|dx| = \,div\,(a^2 \nabla S)\,\,|dx| = 0,$$

$\Longrightarrow$

$$\mathcal{L}_{X_H}\,(\,a^2\,|dx|\,) = 0, \qquad \text{on } \; M\,.$$

Since
the vector field X_H is tangent to L , (Hamilton-Jacobi theorem), and
the Lie derivation is diffeomorpism invariant,
this equation may be lifted by $\pi|L : L \stackrel{\simeq}{\longrightarrow} M$ to L:

$$\mathcal{L}_{X_H}\,(\,a^2\,|dq|\,) = 0, \qquad \pi(q) = x.$$

By $a^2|dq| = (a|dq|^{1/2})^2$ it is rewritten as

$$\mathcal{L}_{X_H}\,(\,a\,|dq|^{1/2}\,) = 0.$$

$$\mathcal{L}_{X_H}\left(a\,|dq|^{1/2} \right) = 0.$$

We have seen that geometric interpretation of the homogeneous transport equation is summarized to the assertion

"there exists a half density on the Lagrangian submanifold L that is invariant under the hamiltonian vector field X_H ."

3 Geometric quantization of a lagrangian embedding $L \overset{i}{\hookrightarrow} T^*M$

3.1

semi-classical approximation of a hamiltonian system H means the following triplet (L, i, a):

1. (L, i) : Lagrangian immersion such that the image $i(L)$ is contained in the energy level set $H^{-1}(E)$ of Hamiltonian H.
2. a : a half density on L that is invariant under the hamiltonian flow of H.

" Quantization " means to have an approximate solution of the Schrödinger equation $\widehat{H} - E = 0$ from the data (L, i, a) .

That is, starting from the phase function S that satisfies the Hamilton-Jacobi equation and
the half-density $a|dq|^{1/2}$ on the Lagrangian space $L = dS \subset T^*M$,
we construct an approximation of the stationary phase solution

$$\varphi = e^{iS/\hbar}(dS)^* a.$$

of the Schrödinger equation; $(H - E)\varphi = \mathbf{O}(\hbar^2)$.

The classical hamiltonian system ($\mathbf{R}^{2n}$, $H = \frac{p^2}{2m} + \frac{kq^2}{2}$) has a quantization $\varphi = e^{iS/\hbar}(dS)^* a$ when S satisfies the Hamilton-Jacobi equation and a satisfies the homogeneous transport equation.

3.2 Pre-quantization of a projectable Lagrangian

Let M be a manifold. We shall investigate the pre-quantization of a Lagrangian submanifold of the cotangent bundle $T^*M \longrightarrow M$. We denote a tangent vector to T^*M by

$$T_{(m,\alpha)}T^*M = T_mM \oplus T^*_mM \ \ni (t,\xi)$$

The canonical 1-form = Liouville form is a 1-form θ on T^*M , that is defined at $(m,\alpha) \in T^*M$ by

$$\theta_{(m,\alpha)}\left((t,\xi)\right) = \alpha(t)\,, \quad \forall (t,\xi) \in T_{(m,\alpha)}T^*M\,,\ .$$

The canonical 2-form is $\omega = d\theta$.

$L \subset T^*M$ is a Lagrangian submanifold iff $\omega|L = 0$.

Example:

$$\begin{aligned} \theta &= \sum_i p_i dq_i\,, \quad \omega = \sum_i dp_i \wedge dq_i\,, \\ L &= image\, dS, \quad \text{for a } S : M \longrightarrow \mathbf{R}. \end{aligned}$$

In the following we shall consider a *projectable* Lagrangian embedding

$$i : L \hookrightarrow T^*M,$$

that is, $\quad \pi|L : L \xrightarrow{\text{diffeo}} M.$

L need not be exact $L \neq \mathit{image}\, dS$ for a S.

$\omega|L = 0$ implies $\theta|L$ closed on L. So, there is a cover $\exists\{L_k\}$ of L, such that $i^*\theta\,|L_k$ is an exact form on each L_k; i.e.

$\exists\, \phi_k$ on L_k (primitive) such that :

$$d\phi_k = i^*\theta\,|L_k.$$

Remark

When the lagrangian embedding $i : L \hookrightarrow T^*M$ is not projectable, it appears singular points of the projection $\pi_L = \pi \circ i : L \longrightarrow M$.
The critical values of π_L are called caustic points of L.

Maslov and Fedoriuk: Semi-classical approximation in quantum mechanics

Duistermaat and Hörmander: Fourier integral operator 2

Therefore the quantization of $(L_k,\, i\,|L_k, \phi_k)$ is given by the oscillatory function on L_k :

$$I_k \;=\; (\pi_{L_k}^{-1})^*\, e^{i\,\phi_k/\hbar}\,.$$

To have the quantization of $(L, i\,)$ we must patch together I_k and give a well defined "quantization" $I(L, i, \phi\,)$ on M :

$$I(L, i, \phi) \;\sim\; \sum_k\, (\pi_{L_k}^{-1})^*\, e^{i\,\phi/\hbar}\;.$$

A sufficient condition is

$$e^{i\,\phi_k/\hbar} \;=\; e^{i\,\phi_j/\hbar}\,, \qquad \text{on} \qquad L_k \cap L_j\,.$$

That is,

$$\phi_j - \phi_k \in\, 2\pi\hbar\cdot\mathbf{Z} \qquad \text{on} \qquad L_k \cap L_j\,. \tag{3.1}$$

This is the Bohr-Sommerfeld condition.

3.3 Bohr-Sommerfeld condition

Let $L = \cup_k L_k$ be a projectable Lagrangian covering. Then

$\exists \quad \phi_j : L_j \longrightarrow \mathbf{R}$, such that $\quad d\phi_k = i^*\theta \,|L_k \,,.$

and $\{ \lambda_{jk} = \phi_j - \phi_k \}$ defines the Liouville class

$\lambda_{(L,i)} \; = [\lambda_{jk}] \in \check{H}^1(L, \, \mathbf{R})$.(independent of covers)

It is also the De Rham cohomology class

$$[\, i^*\theta] \, \in H^1(L, \mathbf{R})$$

Now if the Liouville class satisfies the Bohr-Sommerfeld condition (**??**):

$$\lambda_{jk} = \phi_j - \phi_k \in \, 2\pi\hbar \cdot \mathbf{Z} \quad \text{on} \quad L_k \cap L_j \,. \tag{3.2}$$

$\Longleftrightarrow$

$$\lambda_{(L,i)} = [\lambda_{jk}] \, \in \check{H}^1(L, \mathbf{R})$$

$\{\phi_j\}_j$ defines a

$$\phi : L \longrightarrow \mathbf{T}_\hbar = \mathbf{R}/\mathbf{Z}_\hbar \tag{3.3}$$

such that $i^*\theta \; = \; d\phi$. Thus we have global oscillatory function on L :

$$e^{i\phi/\hbar} \,.$$

3.4 BS-condition as a $\mathbf{T}_{\hbar}$-principal bundle over $L \overset{i}{\hookrightarrow} T^*M$

$\mathbf{T}_{\hbar} = \mathbf{R}/2\pi\hbar\,\mathbf{Z}$: a torus with period $2\pi\hbar$.
We have studied almost the relations:
"Quantization"$\sim$ oscilatory funcrion $\sim$ a section of $\mathbf{T}_{\hbar}$-bundle over the Lagrangian embedding $L \overset{i}{\hookrightarrow} T^*M$.
We shall state them precisely.

θ: the canonical 1-form on T^*M,
$\omega = d\theta$: the symplectic form on T^*M,
$L = \cup_j L_j$, : Lagrangian covering of $L \subset T^*M$
$\phi_j : L_j \longrightarrow \mathbf{R}, \quad d\phi_j = i^*\theta|L_j$
$\lambda_{jk} = \phi_j - \phi_k$: $\quad \lambda_{ij} + \lambda_{jk} + \lambda_{ki} = 0.$

[BS-condition] $\lambda_{jk} \in 2\pi\hbar\,\mathbf{Z}$.

$\mathbf{T}_{\hbar}$-principal bundle $P_{\hbar} \overset{\pi}{\longrightarrow} L$ with the transition functions $\{\lambda_{jk}\}$:

$$(x,t) \sim (x,\, t + \lambda_{jk}(x)\,), \qquad \forall x \in L_j \cap L_k,\, t \in \mathbf{T}_{\hbar}$$

Put $\gamma_j = d\phi_j = i^*\theta|L_j$, then $\{\gamma_j\}_j$ gives a flat connection on the principal bundle $P_{\hbar}$. In fact

$$\gamma_j = \gamma_k + d\lambda_{jk}$$

the curvature : $d\gamma_j = d(i^*\theta|L_j = i^{ast}d\theta|L_j = i^{ast}\omega|L_j = 0$

Bohr-Sommerfeld condition $\lambda_{(L,i)} \in H^1(M, 2\pi \hbar \mathbf{Z})$ implies

$$\phi_j \equiv \phi_k \qquad \bmod \ \mathbf{Z}_\hbar = 2\pi \hbar \mathbf{Z}.$$

So we have a $\mathbf{T}_\hbar = \mathbf{R}/\mathbf{Z}_\hbar$ -valued global section

$$\phi : L \longrightarrow \mathbf{T}_\hbar.$$

(a parallel lift of $P_\hbar \longrightarrow L$).

Let $\rho : \mathbf{T}_\hbar \ni x \longrightarrow e^{-ix/\hbar} \in U(1)$ be a representation of $\mathbf{T}_\hbar$. Associated to the principal bundle $i^*P_\hbar$, we have a line bundle (pre quantum line bundle)

$$\mathcal{E}_\hbar = i^*P_\hbar \otimes_\rho \mathbf{C} \ :$$

The parallel lift $\phi : L \longrightarrow \mathbf{T}_\hbar$ induces a section

$$e^{i\,\phi/\hbar} \ : \ L \longrightarrow U(1)$$

of the line bundle $\mathcal{E}_\hbar$. This is nothing but a global oscillatory function which we were looking for.

Theorem 3.1. *A projectable Lagranjian submanifold $L \hookrightarrow T^*M$ is quantifiable if the correspondding Liouville class is $2\pi\,\hbar$-integral for some $\hbar \in \mathbf{R}_+$.*

The discussion hithereto suggests the following

Abstract Formulation [Kostant, Souriau, Kirilov, Guillemin]

$\bullet$*Pre-quantization of a manifold endowed with a closed 2-form .*

For a manifold X endowed with a closed 2-form σ,
we call a *pre-quantization* of (X, σ)
a hermitian line bundle $(\mathcal{L}\,, <\,, >)$ over X equiped with
a hermitian connection ∇ whose curvature is σ.